ON ARC AND BALL PAIRS AND QUASI-TRANSLATIONS OF THE 3-SPHERE

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INTRODUCTION

All manifolds considered in this paper are not only orientable but already oriented. Let a be an arc in a 3-ball B such that only the end point p(a) (but not the initial point) of a is on the boundary of B and a is locally tame in B except at the end point. Such an *arc and ball pair* will be denoted by (a, B). Two such pairs (a_1, B_1) and (a_2, B_2) are called *equivalent* if there exists an orientation-preserving homeomorphism of B_1 onto B_2 that carries a_1 onto a_2. In this paper, we do not always distinguish between a pair (a, B) and its equivalence class.

Next, we introduce a binary operation # for two arc and ball pairs (a_1, B_1) and (a_2, B_2), and we call it the *composition*. Under the composition #, the set of all equivalence classes of arc and ball pairs forms a semigroup whose identity is the trivial pair (e, B). Further, if $(a_1, B_1) \# (a_2, B_2) = (e, B)$, then

$$(a_1, B_1) = (a_2, B_2) = (e, B)$$

(Theorem 1).

The purpose of introducing such arc and ball pairs (a, B) is to apply them to the construction of quasi-translations (defined below) of the 3-sphere S^3. Let H be an infinite cyclic covering transformation group acting on 3-space E^3, where each element of H is orientation-preserving and the decomposition space E^3/H is a Hausdorff space (an orientable 3-manifold). A generator of H has been called a *quasi-translation of* $S^3 = E^3 \cup (p)$. In other words, h is an orientation-preserving autohomeomorphism of S^3 such that $h(p) = p$ for some point $p \in S^3$, and for every compact subset $C \subset S^3 - (p)$, we have the relation $h^n(C) \cap C = \emptyset$ for all but a finite number of integers n (Sperner's condition). (See [10]. For equivalent definitions see [9].)

In this paper, we construct a quasi-translation h(a, B) of S^3 for each arc and ball pair (a, B), and among other results, we prove that h(a, B) is topologically equivalent to a standard translation if and only if (a, B) is the trivial pair (e, B) (Theorem 5). Applying this, we answer the first problem in [8] in the negative.

In a later paper [13], it will be proved that (i) the composition # is noncommutative in general (hence, by Theorem 2 in this paper, the correspondence $(a, B) \to h(a, B)$ is not one-to-one) and (ii) there exist uncountably many mutually topologically inequivalent quasi-translations of S^3.

It is known that every quasi-translation of the 2-sphere S^2 is topologically equivalent to a standard translation of S^2 (see B. v. Kerékjártó [6], [7] and E. Sperner [14]). The existence of a nontrivial quasi-translation of S^3 was first proved in [9].

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