NORMAL DECOMPOSITIONS IN SYSTEMS WITHOUT THE ASCENDING CHAIN CONDITION

E. W. Johnson and J. P. Lediaev

It is well known that if the family of ideals of a commutative ring satisfies the ascending chain condition (ACC), then every ideal has a finite primary representation [8]. This result has been generalized to various algebraic systems. In particular, the existence of primary representations was investigated in noncommutative rings by H. Tominaga [6]; in noncommutative rings that satisfy the ACC by J. A. Riley [5] and by W. E. Barnes and W. M. Cunnea [1]; and in lattices (with multiplication) that satisfy the ACC by M. Ward and R. P. Dilworth [7] and by R. P. Dilworth [2]. As in the case of noncommutative rings (D. C. Murdoch [4]), the ACC is not sufficient to guarantee the existence of primary representations in (commutative) multiplicative lattices [7]. It is, however, included in each set of sufficient lattice conditions given in [2] and [7]. In particular, Dilworth proved that if L is a modular, multiplicative lattice in which the ACC holds and in which each element is a join of meet-principal elements, then every element in L has a normal decomposition.

In this paper, we consider conditions weaker than the ascending chain condition in order to derive necessary and sufficient criteria for the existence of normal decompositions in multiplicative lattices. We do not use modularity, the existence of meet-principal elements, the ascending chain condition, or the condition that irreducible elements are primary. Since the lattice of ideals of a commutative ring (with an identity) is a multiplicative lattice, our results also extend the normal decomposition theory of Noetherian rings to a wider class of rings.

A *commutative, multiplicative lattice* is a complete lattice in which there is defined a commutative, associative, and join-distributive multiplication for which the greatest element, denoted by I, is the multiplicative identity. For basic properties of such lattices, see [2].

Throughout this paper, L will denote a commutative, multiplicative lattice. We say that L satisfies the *successive residual condition* if, for each element A in L, chains of the form

$$A:C_1 \leq A:(C_1 C_2) \leq \cdots \leq A:(C_1 C_2 \cdots C_n) \leq \cdots$$

are finite. This condition was used by W. Krull [3] to study minimal primes and associated primes of ideals. An element C ($C \neq I$) is the *residual component of* A *by* B (or simply a residual component of A) if there exists a positive integer n such that

$$C = A:B^n = A:B^{n+1} = \cdots .$$

We note that if A has a normal decomposition, then the residual components of A are precisely the isolated components of A. If each element A in L has only a finite number of distinct residual components, we say L satisfies the *residual-component*

Received February 3, 1969.

Michigan Math. J. 17 (1970).