ON TWO SUM THEOREMS FOR IDEALS OF C(X)

David Rudd

1. INTRODUCTION

Let C(X) denote the ring of all continuous real-valued functions on a completely regular Hausdorff space X. It is well known (see [1, p. 198]) that in C(X) the sum of two z-ideals is a z-ideal and the sum of two prime ideals is a prime ideal.

L. Gillman and C. W. Kohls have remarked [2, p. 401] that the proofs of these assertions seem to depend strongly on properties of βX, the Stone-Čech compactification of X. The purpose of this note is to present elementary proofs of both theorems without using any properties of βX.

To emphasize that βX plays no apparent role, we prove the assertions for ideals of subrings of C(X), provided these subrings are absolutely convex sublattices of C(X). The methods of [1] do not seem to yield the sum theorem for z-ideals in such subrings.

2. PRELIMINARIES

An ideal I of a commutative ring R is said to be *semiprime* in R if, for each $x \in R$, we have that $x \in I$ whenever $x^2 \in I$. It is well known (see [1, p. 31]) and easy to prove that the semiprime ideals of a commutative ring are precisely the intersections of prime ideals.

A subring $\mathscr{A}$ of a lattice-ordered ring R is said to be *absolutely convex* in R if $x \in R$, $y \in \mathscr{A}$, and $|x| \leq |y|$ imply $x \in \mathscr{A}$. For the remainder of this note, let $\mathscr{A}$ denote some absolutely convex subring of C(X).

We remark that if f is an element of $\mathscr{A}$, then $|f|$ is an element of $\mathscr{A}$, since $|(|f|)| \leq |f|$.

We denote by Z(f) the set of all $x \in X$ such that $f(x) = 0$.

LEMMA 2.1. *A prime ideal* P *in* $\mathscr{A}$ *is absolutely convex in* $\mathscr{A}$.

Proof. Let $f \in \mathscr{A}$, $p \in P$, and suppose that $|f| \leq |p|$. Define g as in 5.5 of [1]; that is, let

$$g = \begin{cases} 0 & \text{on } Z(p), \\ \dfrac{f^2}{|p|} & \text{on } \sim Z(p). \end{cases}$$

Then g is easily seen to be continuous, and the inequality

$$|g| \leq \frac{|f| \cdot |f|}{|p|} \leq |f|$$

Received December 16, 1969.
This work was done while the author was on a NASA graduate fellowship.

Michigan Math. J. 17 (1970).