THE HAUSDORFF DIMENSION OF CERTAIN SETS OF NONNORMAL NUMBERS

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1. THE GENERATION OF MEASURES

Let $C[0, 1)$ denote the collection of continuous, periodic functions (with period 1) on the reals, and let s be a fixed integer ($s \geq 2$). We say that a real number x is *ν-normal* (to the base s) if the sequence $\{s^n x\}$ has the distribution ν in the sense that

$$(1.1) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(s^k x) = \int_0^1 f\,d\nu = \nu(f) \qquad \text{for each } f \in C[0, 1).$$

Here, ν denotes a probability measure on $[0, 1)$ that is invariant under the transformation $T: y \to Ty = sy - [sy]$. In other words, ν is the weak* limit of the measures μ_n defined by the relations

$$(1.2) \qquad \mu_n(f) \equiv \mu_{n,x}(f) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x).$$

We say that x *generates* ν (to the base s). The space of all these T-invariant probability measures, together with the weak* topology, is denoted by $I(s)$. Observe that for each positive integer n we have the inclusion $I(s) \subset I(s^n)$. One measure in $I(s)$ is the ordinary Lebesgue measure λ, and x is normal to the base s in the classical sense precisely when x is λ-normal to the base s.

Consider a measure ν in $I(s)$, and let x generate ν (to the base s). As is well known (see for instance [5]), relation (1.1) also holds when f is bounded and the set of discontinuities of f has ν-measure 0. In particular, the relation (1.1) holds when f is the characteristic function of an interval $[\alpha, \beta)$. That is,

$$(1.3) \qquad \lim_{n \to \infty} \frac{1}{n} (\text{number of } k \ (0 \leq k \leq n-1), \text{ for which } T^k x \in [\alpha, \beta))$$
$$\equiv \lim_{n \to \infty} \mu_n([\alpha, \beta)) = \nu([\alpha, \beta)),$$

provided $\nu(\{\alpha\}) = 0 = \nu(\{\beta\})$. Also, it is known that if the relation (1.3) holds for some point x and for all choices of α and β such that $\nu(\{\alpha\}) = 0 = \nu(\{\beta\})$, then x is ν-normal (to the base s) (see [5]). In fact, for x to be ν-normal, it suffices to require that condition (1.3) hold only for intervals of the form $[as^{-n}, (a+1)s^{-n})$,

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