THE K-NULLITY SPACES OF THE CURVATURE OPERATOR

Yeaton H. Clifton and Robert Maltz

1. INTRODUCTION

For any real constant K, set $K_{xy}(z) = R_{xy}(z) - K\{\langle x, z\rangle y - \langle y, z\rangle x\}$, where R denotes the curvature tensor, $\langle\ ,\ \rangle$ denotes the Riemannian inner product, and x, y, z belong to the tangent space M_m of the Riemannian manifold M, at the point m. Let $N_K(m) = \{x \in M_m \mid K_{xy} = 0 \text{ for all } y \in M_m\}$. We call $N_K(m)$ the K-*nullity space* at m, and we call $\mu_K(m) = \dim N_K(m)$ the *index of* K-*nullity* at m (T. Ôtsuki [6]).

$N_K(m)$ and $\mu_K(m)$ generalize the concepts of the *nullity space* $N_0(m)$ and of the *index of nullity* $\mu_0(m)$, which constitute the case K = 0. S. S. Chern and N. H. Kuiper [2] showed that N_0 defines an involutive distribution, and that if μ_0 is constant in a neighborhood, then the leaves of the resulting foliation are locally flat in the induced metric. R. Maltz [5] showed the following.

(i) The leaves are actually totally geodesic submanifolds of M (this implies they are locally flat).

(ii) If G denotes the open set on which μ_0 takes its minimum value m_0 (assumed to be positive), and if M is complete, then the leaves of the nullity foliation of G are also complete.

(iii) The nullity distribution N_0 has no isolated singular points (a singular point is a point at which the dimension μ_0 is not locally constant).

(iv) The boundary of G is the union of geodesics tangent to N_0.

Both involution of the distribution and property (i) are local, essentially algebraic results; since K_{xy} satisfies precisely the same algebraic conditions as R_{xy} (Ôtsuki [6]), it is obvious that N_K is involutive and has property (i) for all K (A. Gray [3]). It follows, of course, that the leaves of the foliation (for locally constant μ_K) have constant curvature K.

Properties (ii), (iii), and (iv), on the other hand, are global results. It is the purpose of this paper to establish them for arbitrary K. The essential idea is contained in the following result.

THEOREM (*). *Let* M *be a complete Riemannian manifold. Suppose* G *is an open subset of* M *on which the* K-*nullity index* μ_K *takes the constant value* m. *If* L *is a leaf of the* K-*nullity foliation induced on* G, *and if* $\gamma[0, c)$ *is a geodesic segment lying in* L, *then* $\lim_{t \to c^-} \gamma(t)$ *lies in* L *also.*

Remarks. (1) μ_K is easily seen to be upper-semicontinuous; therefore the set G on which μ_K attains its minimum value m_K is open. If $m_K > 0$, we actually obtain a foliation of G.

(2) It is easy to verify, by a simple generalization of Schur's Theorem, that no further generality can be obtained by allowing K to vary from point to point, except in the case where $\mu_K = 1$.

Received January 20, 1968.

Michigan Math. J. 17 (1970).