STRONG INERTIAL COEFFICIENT RINGS

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INTRODUCTION

A well-known theorem due to J. H. M. Wedderburn and A. Malcev states that if A is a finite-dimensional algebra over a field F and if A/N is separable over F, then there exists a subalgebra S of A such that

$$S + N = A \quad \text{and} \quad S \cap N = 0.$$

Here N denotes the radical of A. The theorem further states that S is unique up to an inner automorphism G of A. The automorphism G is of the form

$$G(X) = (1 - n)X(1 - n)^{-1},$$

for some n in N. Many authors have attempted to generalize this result by removing the restriction that F be a field. In particular, G. Azumaya [3] has extended the Wedderburn-Malcev theorem to the case where F is a Hensel ring. Azumaya proved the following result: Let A be an algebra over a Hensel ring R. If A is finitely generated as an R-module and if A/N is separable over R/p, then A contains a subalgebra S that is separable over R and has the property that S + N = A. Here N and p are the Jacobson radicals of A and R, respectively. Azumaya further proved that S is unique up to an inner automorphism G of A, where G is as in the original classical theorem. If R is a field, then $S \cap N = 0$, and we retrieve the original theorem. The Wedderburn-Malcev theorem yields an F-algebra isomorphism of A/N into A. Since $S \cap N \neq 0$ in general, we lose this isomorphism in Azumaya's generalization.

In [6], E. Ingraham has studied a class of commutative rings, called inertial coefficient rings, that permit a generalization of the Wedderburn-Malcev theorem along the lines of Azumaya's result. Specifically, a commutative ring R with identity is called an *inertial coefficient ring* if it has the following property: If A is an R-algebra that is finitely generated as an R-module and has the property that A/N is separable over R, then there exists an R-separable subalgebra S of A with S + N = A. If S is unique up to an inner automorphism of A generated by 1 plus an element of N, then R is said to have the *uniqueness property*. In these terms, Azumaya's result says that every Hensel ring is an inertial coefficient ring with the uniqueness property.

If A is an algebra over an inertial coefficient ring R and A satisfies the usual hypotheses, then there need not exist an algebra isomorphism of A/N into A that splits the sequence

$$0 \to N \to A \to A/N \to 0.$$

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