LOCALLY COMPACT, TOTALLY DISCONNECTED, SOLVABLE GROUPS

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In this note, we continue our study of the structure of locally compact, totally disconnected groups [1]. Let G be a topological group, and let the inner automorphisms act naturally on G. An element g in G has a relatively compact orbit if the closure of the orbit of g under the actions of inner automorphisms is compact. As we know, the set P(G) of compact elements in a locally compact, totally disconnected, solvable group G need not be a subgroup of G. We shall prove, however, that the set $P(G) \cap B(G)$ is a characteristic subgroup of G, where

$$B(G) = \{g \mid g \text{ has relatively compact orbit}\}.$$

We shall apply this result to obtain information about the structure of a solvable group G that has a finitely-generated, free abelian subgroup Z^n such that G/Z^n is compact.

From now on, G denotes a locally compact, totally disconnected topological group with identity e. Let B(G) denote the subset of G consisting of all points whose orbits are relatively compact under the action of the group of inner automorphisms of G; in other words, $B(G) = \{x \in G \mid \overline{I_G(x)} \text{ is compact}\}$, where $I_G(x) = \{gxg^{-1} \mid g \in G\}$. An element x in G is a compact element if x is contained in a compact subgroup of G. Let

$$P(G) = \{x \mid x \in G, \ x \text{ is a compact element}\}.$$

Let H be a topological group, and let H' denote the closure of the commutator (derived) subgroup of H. A topological group H is solvable if there exists an integer n such that

$$H^0 = H, \quad H^{i+1} = (H^i)' \quad (0 \leq i \leq n-1), \quad \text{and } H^n = \{e\}.$$

Remark 1. A locally compact, totally disconnected group contains compact-open subgroups; but, in general, even a nilpotent group need not contain a compact-open, normal subgroup. It is known that G contains compact-open, normal subgroups if and only if B(G) is an open subgroup of G [1].

Remark 2. In general, P(G) does not form a subgroup. The group G generated by two elements a, b satisfying the relation $a^2 = b^2 = e$, together with the discrete topology, provides an example. (The group G can also be obtained as the semidirect product of the integers Z with the automorphism $\theta: Z \to Z$, defined by $\theta(n) = -n$.)

PROPOSITION 1. *Suppose* G *is a solvable group. Then* $R(G) = P(G) \cap B(G)$ *is a subgroup of* G.

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