CONCORDANCE CLASSES OF SPHERE BUNDLES OVER SPHERES

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The purpose of this paper is to provide a proof for a theorem announced in [5] concerning the classification, up to concordance, of differentiable structures on manifolds that are sphere bundles over spheres. This is finer than classification up to diffeomorphism; R. DeSapio [1] has proved results on the latter problem that are interesting to compare with ours.

We recall that a *concordance* between two differentiable structures β and β' on a nonbounded PL (piecewise-linear) manifold K is a differentiable structure γ on the PL manifold $K \times I$ that equals β on $K \times 0$ and β' on $K \times 1$. If we denote the set of equivalence classes under the relation of concordance by C(K), the theorem in question may be stated as follows.

THEOREM. *Let* K *be the total space of an* S^j*-bundle over* S^i *whose characteristic map may be pulled back to an element* α *of* $\pi_{i-1}(SO(j))$. *Then there exists a one-to-one correspondence*

$$C(K) \longleftrightarrow \Gamma_i \oplus A \oplus [\Gamma_{i+j}/\text{image } \tau_\alpha],$$

where A *is a subgroup of* Γ_j. *If* α *can be pulled back to an element* α' *of* $\pi_{i-1}(SO(j-1))$, *then* $A = \Gamma_j \cap (\text{kernel } \tau_{\alpha'})$.

Here Γ_n denotes the group of diffeomorphisms of S^{n-1}, modulo the subgroup consisting of those diffeomorphisms extendable to B^n. It is isomorphic with $C(S^n)$, the operation being connected sum. For $n \geq 5$, $C(S^n)$ is equal to the group Θ_n of (oriented diffeomorphism classes of) differentiable structures on S^n. The group Γ_n is abelian and finite; it vanishes for $n \leq 6$.

The maps $\tau_{\alpha'}\colon \Gamma_j \to \Gamma_{i+j-1}$ and $\tau_\alpha\colon \Gamma_{j+1} \to \Gamma_{i+j}$ are the so-called "Milnor-Munkres-Novikov" twisting homomorphisms. (See [5, p. 189], where the homomorphism

$$\tau\colon \pi_k(SO(m-1)) \otimes \Gamma_m \to \Gamma_{m+k}$$

is defined; in the present paper we denote $\tau(\alpha, x)$ by $\tau_\alpha(x)$.)

Examples. Suppose K is the nontrivial S^j-bundle over S^2 $(j > 1)$; we compare its concordance classes with those of the trivial bundle $S^j \times S^2$. Of course, C(K) is never larger than $C(S^j \times S^2)$, since $\tau_\alpha = 0$ if $\alpha = 0$; and there exist many values of j for which C(K) is strictly smaller, for example, j = 7, 13, 14, 15, and $j \equiv 0$ or $j \equiv 1$ modulo 8. This follows from the fact that if $\alpha(k)$ is the nontrivial element of $\pi_1(SO(k))$ $(k > 2)$, then $\tau_{\alpha(k)}\colon \Gamma_{k+1} \to \Gamma_{k+2}$ is nontrivial for k = 7, 13, 15 and for $k \equiv 0 \pmod 8$. (See J. Levine [4], noting that his $\delta(\sigma, 0; 0, \alpha)$ is just our $\tau_\alpha(\sigma)$.)

As a second example, take K to be an S^j-bundle over S^i having two independent cross-sections (so that α' exists), where j is fairly close to i $(1 \leq i - 3 \leq j \leq i + 1)$;

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