PRIMES IN ARITHMETIC PROGRESSIONS

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The purpose of this paper is to investigate the asymptotic size of the quantity

$$(1) \qquad \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \psi(x; q, a) - \frac{x}{\phi(q)} \right|^2,$$

where

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a (q)}} \Lambda(n).$$

In recent years, a number of mathematicians have used the large sieve to obtain upper bounds for (1) when Q is slightly smaller than x. M. B. Barban [1] seems to have been the first, and he was closely followed by H. Davenport and H. Halberstam [6], who showed that (1) is $\ll Qx(\log x)^5$ (Vinogradov's notation) for $Q \leq x$. Later, P. X. Gallagher [8] introduced some simplifications and made more precise estimates to show that (1) is $\ll Qx \log x$, and R. J. Wilson used the large sieve in an algebraic number field to obtain similar results in algebraic number fields [12]. In this paper we show, *without using the large sieve*, that these bounds may be replaced by an asymptotic equality with an explicit error term.

THEOREM. *Let* A *be fixed* $(A > 0)$. *For* $Q \leq x$,

$$(2) \qquad \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \psi(x; q, a) - \frac{x}{\phi(q)} \right|^2 = Qx \log x + O\left(Qx \log \frac{2x}{Q}\right) + O(x^2 (\log x)^{-A}),$$

and for $Q \geq x$,

$$(3) \qquad \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \psi(x; q, a) - \frac{x}{\phi(q)} \right|^2 = Qx \log x - \frac{\zeta(2)\, \zeta(3)}{\zeta(6)} x^2 \log \frac{Q}{x} - Qx + A_1 x^2 + O(Qx (\log x)^{-A}).$$

The first error term in (2) may no doubt be reduced, but it appears that the error is genuinely $\gg Qx \log \log x/Q$. Halberstam has pointed out that in an obscure journal Barban [2] asserted (3) in the case $x = Q$. However, he seems not to have indicated what lines the proof was to take, and it may now be impossible ever to determine what he had in mind.

We note that our method suggests that perhaps

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