A COUNTEREXAMPLE TO A CONJECTURE IN SECOND-ORDER LINEAR EQUATIONS

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Consider the differential equation

$$u'' + a(t)u = 0, \tag{1}$$

where $a(t)$ is a positive, nondecreasing, unbounded function in $C'[T, \infty)$. It is well known that the hypotheses on $a(t)$ do not imply that every solution of (1) satisfies the condition

$$u(t) \to 0 \quad \text{as } t \to \infty. \tag{2}$$

L. A. Gusarov [3] has shown that under the additional hypothesis that $a'(t)$ is of bounded variation on $[T, \infty)$, the solutions of (1) satisfy condition (2). Under these assumptions, $a'(t)$ has a finite, nonnegative limit as $t \to \infty$. A. Meir, D. Willett, and J. S. W. Wong [4] have proved the following result.

THEOREM 1. *If there exists a positive function* $p(t) \in C'[0, \infty)$ *such that*

$$\int_0^\infty \frac{dt}{p(t)} = +\infty, \qquad \liminf_{t\to\infty} \frac{p'(t)}{p(t)\,a^{1/2}(t)} \geq 0, \qquad \textit{and} \ \liminf_{t\to\infty} \frac{a'(t)\,p(t)}{a(t)} > 0,$$

then the solutions of (1) *satisfy condition* (2).

From this result it follows that if $a'(t)$ is ultimately bounded and bounded away from zero, then all solutions of (1) satisfy (2). The following question presents itself: does the condition that $a'(t) \to 0$ as $t \to \infty$ (or that $\limsup a'(t) < \infty$) imply that condition (2) holds for all solutions of (1)? Meir, Willett, and Wong [4] conjectured that if in Theorem 1 the last condition is replaced by the condition

$$\lim_{t\to\infty} a'(t)\,p(t)/a(t) = 0,$$

then the conclusion remains valid. If this conjecture were true, we could answer our question in the affirmative (simply set $p(t) \equiv 1$). However, the following theorem shows that the conjecture is false.

THEOREM 2. *For each* $\beta > 0$, *there exists a positive function* $a(t) \in C^\infty[0, \infty)$ *such that* $a(t) \to \infty$, $a'(t) \geq 0$, $a'(t) = o(\log^{-\beta} t)$, *and such that at least one solution* $u(t)$ *of* (1) *satisfies the condition* $\limsup_{t\to\infty} |u(t)| > 0$.

Without loss of generality, we replace the condition $a'(t) = o(\log^{-\beta} t)$ by $a'(t) = O(\log^{-m} t)$, where m is an integer $(m > \beta)$. The proof is based on a method

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