ZEROS OF PARTIAL SUMS OF POWER SERIES. II

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1. INTRODUCTION

Problem 7.7 in W. K. Hayman's *Research Problems in Function Theory* [3] is the following: Let $f(z) = \sum_0^\infty a_k z^k$ denote an analytic function whose power series has radius of convergence 1. Set

$$S_n(z) = S_n(z; f) = \sum_{k=0}^{n} a_k z^k \qquad (n = 1, 2, 3, \cdots),$$

and let $\rho_n(f)$ denote the largest of the moduli of the zeros of S_n (with the convention that $\rho_n(f) = \infty$ if $a_n = 0$). Let

$$\rho(f) = \liminf_{n \to \infty} \rho_n(f)$$

and

$$P = \sup_f \rho(f).$$

The problem is to determine the value of P. In [2], J. Clunie and P. Erdös showed that $\sqrt{2} < P < 2$. The present author [1] obtained the estimates $1.7 < P \leq 12^{1/4}$. Later, J. L. Frank [1] improved these bounds to $1.7818 < P < 1.82$.

In the present paper, I determine the exact value of P. The determination depends on certain algebraic relations between the coefficients of a power series and the zeros of its partial sums. These relations are most conveniently expressed in terms of the polynomials $B_n(z; z_0, \cdots, z_{n-1})$ defined by

$$(1.1) \qquad B_0(z) = 1, \qquad B_n(z; z_0, \cdots, z_{n-1}) = z^n - \sum_{k=0}^{n-1} z_k^{n-k} B_k(z; z_0, \cdots, z_{k-1}).$$

(Here $B_k(z; z_0, \cdots, z_{k-1})$ is to be interpreted as 1 when $k = 0$.)

Set

$$H_n = \max |B_n(0; z_0, \cdots, z_{n-1})| \qquad (n = 0, 1, 2, \cdots),$$

where the maximum is taken over all sequences $\{z_k\}_0^{n-1}$ whose terms lie on the unit circle. On the basis of the algebraic relations mentioned above, we obtain the following result.

THEOREM 1. $P = \sup_{1 \leq n < \infty} H_n^{1/n} = \lim_{n \to \infty} H_n^{1/n}$.

Received March 4, 1969.

This research was supported in part by NSF Grant GP-8225 and Basic Science Development Grant.

Michigan Math. J. 17 (1970).