ON A VARIATIONAL METHOD FOR UNIVALENT FUNCTIONS

Ch. Pommerenke

There are several proofs for the basic results on interior variations of univalent functions. The original proof of M. Schiffer [5] uses a variation of the Green's function and an approximation of the domain by smooth curves. The proof of G. M. Golusin [2], [3, p. 96] applies only to analytic variations (which are sufficient for almost all applications), and it uses the majorant method. For further discussions of this variational method, see [1], [4], [6].

We give a new proof of Golusin's version of the variational theorem. The proof is elementary, apart from the use of Carathéodory's kernel theorem. By univalent we mean analytic and univalent.

THEOREM. *Let* $f(z)$ *be univalent in* $|z| < 1$ *and normalized so that* $f(0) = 0$ *and* $f'(0) > 0$. *For* $0 < \lambda < \lambda_0$, *let* $g(z, \lambda)$ *be univalent in the annulus* $r < |z| < 1$, *where* r *is some fixed number. Let*

$$\frac{g(z, \lambda) - f(z)}{\lambda z f'(z)} \to \sum_{n=1}^{\infty} c_{-n} z^{-n} + c_0 + \sum_{n=1}^{\infty} c_n z^n \qquad (\lambda \to 0+), \tag{1}$$

locally uniformly in $r < |z| < 1$.

For $0 < \lambda < \lambda_0$, *let the univalent function* $f(z, \lambda)$ *map* $|z| < 1$ *onto the union of the doubly connected domain* $\{g(z, \lambda): r < |z| < 1\}$ *and the compact set enclosed by this domain, and let* $f(0, \lambda) = 0$ *and* $f'(0, \lambda) > 0$. *Then*

$$\frac{f(z, \lambda) - f(z)}{\lambda z f'(z)} \to \Re c_0 + \sum_{n=1}^{\infty} (c_n + \bar{c}_{-n}) z^n \qquad (\lambda \to 0+), \tag{2}$$

locally uniformly in $|z| < 1$.

Remark. The choice

$$g(z, \lambda) = f(z) + \frac{a\lambda f(z)^2}{f(z) - f(z_0)} \qquad (|z_0| < 1,\ |a| = 1,\ 0 < \lambda < \lambda_0)$$

leads to a special case of Schiffer's variational formula [5].

If S is the usual class of normalized univalent functions and $f(z)$ belongs to S, it follows from (2) that the function

$$f^*(z, \lambda) = f(z) + \left[(z f'(z) - f(z)) \Re c_0 + z f'(z) \sum_{n=1}^{\infty} (c_n + \bar{c}_{-n}) z^n \right] \lambda + o(\lambda)$$

belongs to S and is a variation of the function $f(z)$.

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