SINGULAR INNER FACTORS OF ANALYTIC FUNCTIONS

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If the analytic function f is in a Hardy class H^p $(p > 0)$, then f has a factorization $f = BSF$, where B is a Blaschke product, F is an outer function in H^p, and S, the *singular factor* of f, has the form

$$S(z) = \exp\left\{-\int \frac{e^{it} + z}{e^{it} - z}\, d\mu(e^{it})\right\},$$

where μ is a positive singular measure on the unit circle. The product BS is called the *inner factor* of f. See Chapter 5 of K. Hoffman's book [2] for a discussion.

If f is not constant, then the set of complex numbers c such that $f - c$ has a nonconstant inner factor is uncountable; indeed, if c is in the range of f, then $f - c$ has a nontrivial Blaschke factor. The situation is different if we restrict our attention to singular factors. We have the following result.

THEOREM 1. *If* $f' \in H^1$, *then* $f - c$ *has a nonconstant singular factor for at most countably many values of* c.

The proof requires the following two results.

LEMMA 1. *Let* $\{\mu_\alpha\}$ *be a family of positive Borel measures on the unit circle with disjoint supports. If there exists a finite positive Borel measure* μ *such that* $\mu - \mu_\alpha \geq 0$ *for all* α, *then at most countably many of the measures* μ_α *are nonzero.*

The proof is easy. The proof of the next result appears in [1, Theorem 1] (see the note added in proof).

LEMMA 2. *If* $f' \in H^1$, *then the singular factor of* f *divides* f'.

Proof of Theorem 1. Since $f' \in H^1$, the function f is continuous on the closed unit disc (see [2, p. 70]). Let μ_c denote the singular measure associated with the singular factor of $f - c$. By a result of W. Rudin (see [4, Lemma 6]), $f - c$ must vanish on the support of μ_c. Hence the measures $\{\mu_c\}$ have disjoint supports. But by Lemma 2, the singular factor of $f - c$ must divide f', and hence it must divide the singular factor of f'. Thus, if μ is the singular measure associated with the singular part of f', then $\mu - \mu_c \geq 0$ for all c. The result now follows from Lemma 1.

If we drop the requirement that $f' \in H^1$ and require merely that $f \in H^\infty$, then the conclusion of the theorem need not hold.

THEOREM 2. *There exists an inner function* ϕ *such that* $\phi - c$ *has a nonconstant singular factor for uncountably many values of* c.

Proof. Let ϕ be some inner function that omits an uncountable set of values in the unit disc (for the existence of such functions see [3, Theorem 12 in Section II.4 and Footnote 3 on page 26]). Let c be an omitted value. Then

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