LOCAL BOUNDEDNESS OF NONLINEAR, MONOTONE OPERATORS

R. T. Rockafellar

1. INTRODUCTION

Let X denote a locally convex Hausdorff (topological vector) space over the reals R. Let X^* denote the dual of X, and write $\langle x, x^* \rangle$ in place of $x^*(x)$ for $x \in X$ and $x^* \in X^*$.

A multivalued mapping $T: X \to X^*$ is called a *monotone operator* if

$$(1.1) \qquad \langle x - y, x^* - y^* \rangle \geq 0$$

whenever $x^* \in T(x)$ and $y^* \in T(y)$. It is called a *maximal* monotone operator if, in addition, the graph of T, in other words, the set

$$(1.2) \qquad \{(x, x^*) \mid x^* \in T(x)\} \subset X \times X^*,$$

is not properly contained in the graph of any other monotone operator $T': X \to X^*$. It is said to be *locally bounded* at x if there exists a neighborhood U of x such that the set

$$(1.3) \qquad T(U) = \bigcup \{T(y) \mid y \in U\}$$

is an equicontinuous subset of X^*. (Of course, if X is a Banach space, then the equicontinuous subsets of X^* coincide with the bounded subsets.)

In the case where X is a Banach space, it follows from a result of T. Kato [7] that a monotone operator $T: X \to X^*$ is locally bounded at a point x if x is an interior point of the set

$$(1.4) \qquad D(T) = \{x \in X \mid T(x) \neq \emptyset\}$$

and T is locally hemibounded at x (in other words, for each $u \in X$ there exists an $\varepsilon > 0$ such that the set

$$\bigcup \{T(x + \lambda u) \mid 0 \leq \lambda \leq \varepsilon\}$$

is equicontinuous in X^*). Moreover, Kato showed in [6] that the assumption of local hemiboundedness is redundant when X is finite-dimensional.

In this note, we establish the following more general result, which implies, among other things, that the assumption of local hemiboundedness is redundant even when X is an infinite-dimensional Banach space. (The abbreviations conv, int, and cl denote convex hull, interior, and (strong) closure, respectively.)

Received September 18, 1968.
This work was supported in part by the Air Force Office of Scientific Research under grant AF-AFOSR-1202-67.