FUNCTIONS SATISFYING LIPSCHITZ CONDITIONS

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In memory of Professor A. Robert Brodsky (1940-1968)

Let (X, d) be a metric space, and let $\alpha > 0$. A real-valued function f on X is said to be of Lipschitz class α if

$$\sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \,\middle|\, x, y \in X, x \neq y \right\}$$

is finite. The purpose of this paper is to investigate metric spaces that support nonconstant functions of Lipschitz class α, with emphasis on the case $\alpha > 1$. In addition to investigating the metric spaces themselves, we shall also investigate the structure of various Banach algebras of functions satisfying Lipschitz conditions.

1. A PRELIMINARY PROPOSITION

Throughout the paper, we shall be concerned with real-valued functions on (X, d); if f is of Lipschitz class α, we denote by $\|f\|_\alpha$ the defining supremum. Let $\mathrm{Lip}_\alpha(X, d)$ denote the set of all bounded functions on X of Lipschitz class α; if $f \in \mathrm{Lip}_\alpha(X, d)$, let $\|f\|_\infty = \sup_{x \in X} |f(x)|$. The following proposition is of interest, since the proof differs from the argument in [3].

PROPOSITION 1.1. *For* $f \in \mathrm{Lip}_\alpha(X, d)$, *let* $\|f\| = \|f\|_\alpha + \|f\|_\infty$. *With this norm,* $\mathrm{Lip}_\alpha(X, d)$ *is a Banach algebra.*

Proof. The verification that $\mathrm{Lip}_\alpha(X, d)$ is a normed algebra parallels the argument in [4]; it remains to show $\mathrm{Lip}_\alpha(X, d)$ is complete. Let

$$f_n \in \mathrm{Lip}_\alpha(X, d) \qquad (n = 1, 2, \cdots),$$

and suppose $\|f_n - f_m\| \to 0$; then $\|f_n - f_m\|_\infty \to 0$, and therefore there exists a function $f \in C(X)$ such that $f_n \to f$ uniformly. Now

$$|f(x) - f(y)| \leq |f_n(x) - f_n(y)| + |(f - f_n)(x)| + |(f - f_n)(y)|,$$

and hence, given $\varepsilon > 0$ and $x \neq y$, we can choose N so that

$$\|f - f_N\|_\infty < \frac{\varepsilon}{2} d(x, y)^\alpha.$$

Then

$$|f(x) - f(y)| \leq \|f_N\|_\alpha d(x, y)^\alpha + \varepsilon d(x, y)^\alpha = (\|f_N\|_\alpha + \varepsilon) d(x, y)^\alpha,$$

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