AN ELEMENTARY PROOF OF THE FIXED-POINT THEOREM OF BROWDER AND KIRK

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1. INTRODUCTION

In [1] and [3], F. E. Browder and W. A. Kirk showed independently that a nonexpansive self-mapping of a nonempty, closed, convex set in a uniformly convex Banach space has a fixed point. Their proofs are similar, and both are based on Zorn's Lemma and other nonelementary theorems of functional analysis.

We shall give an elementary proof of this fixed-point theorem, using only the definition of uniform convexity and some basic theorems of topology and analysis.

2. NOTATION AND DEFINITIONS

Let B be a uniformly convex Banach space with norm $\|\ \|$ and zero element Θ. Let K be a nonempty, closed, bounded, convex subset of B, and suppose (without loss of generality) that $\Theta \in K$. Let d(X) denote the diameter of the set $X \subset B$, and set $a(X) = \inf_{x \in X} \|x\|$. Finally, let $I_1 = (0, 1]$ and $I_2 = (0, 2]$.

The following definition of uniform convexity is equivalent to the classical one [2].

Definition 1. The Banach space B is called *uniformly convex* if there exists an increasing, positive function $\delta: I_2 \to I_1$ such that the inequalities $\|x\| \leq r$, $\|y\| \leq r$, and $\|x - y\| \geq \varepsilon r$ imply that

$$\left\|\frac{x+y}{2}\right\| \leq (1 - \delta(\varepsilon))r \qquad (x, y \in B).$$

It is obvious that $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ and $\delta(2) = 1$. We denote the inverse of δ by η, and we observe that $\lim_{y \to 0} \eta(y) = 0$.

Definition 2. A transformation $F: K \to K$ is called *nonexpansive* if the inequality $\|Fx - Fy\| \leq \|x - y\|$ holds for all x and y in K. A transformation F is a *contraction* if there exists a constant k $(0 \leq k < 1)$ such that $\|Fx - Fy\| \leq k\|x - y\|$ for all $x, y \in K$.

3. THE THEOREM OF BROWDER AND KIRK

THEOREM. *Every nonexpansive mapping* $F: K \to K$ *has at least one fixed point.*

LEMMA. *If* u, v, w *are elements of* B *such that*

$$\|u - w\| \leq R, \qquad \|v - w\| \leq R, \qquad \text{and} \left\|w - \frac{u+v}{2}\right\| \geq r > 0,$$

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