ON A NONLINEAR VOLTERRA EQUATION

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1. INTRODUCTION

We study the asymptotic behavior of solutions of the integrodifferential equation

$$(1.1) \qquad x'(t) = -\int_0^t a(t-\tau)g(x(\tau))d\tau - b(t) + f(t) \qquad (0 \le t < \infty)$$

(primes denote differentiation with respect to t), where a(t) satisfies the conditions

$$(1.2) \qquad \begin{array}{c} a(t) \in C(0,\infty) \cap L_1(0,1);\ a(t) \text{ is nonnegative, nondecreasing,} \\ \text{and convex on } (0,\infty);\ \text{and } 0 < a(0+) \le \infty . \end{array}$$

The functions g and f will be subject to the conditions

$$(1.3) \qquad g(x) \in C(-\infty,\infty), \qquad xg(x) \ge 0, \qquad G(x) = \int_0^x g(\xi)d\xi \to \infty \quad (|x| \to \infty)$$

and

$$(1.4) \qquad f(t) \in C[0,\infty), \qquad K_0 = \int_0^\infty |f(t)|\,dt < \infty .$$

We first find conditions ensuring that all solutions x(t) of (1.1) satisfy the condition

$$(1.5) \qquad \lim_{t\to\infty} x(t) = 0 .$$

Our result extends a theorem of J. J. Levin and J. A. Nohel [6, Theorem 1(ii)], which deals with the case where $a(t) \in C[0,\infty)$ and $(-1)^k a^{(k)}(t) \ge 0$ $(0 < t < \infty;\ k = 0, 1, 2, 3)$.

For the linear case $(g(x) = x)$ with $f(t) \equiv 0$ and $b(t) \equiv$ constant, we showed in [3] that there exist kernels a(t), satisfying (1.2), for which a solution x(t) does not satisfy (1.5); indeed there exists a nonconstant periodic function $\omega(t)$ such that $[x(t) - \omega(t)] \to 0$ as $t \to \infty$. These kernels satisfy the equation

$$(1.6) \qquad a(t) = \delta_0 + \sum_{k=1}^{\infty} \delta_k \left(1 - \frac{\min\{t, kt_0\}}{kt_0}\right),$$

Received November 14, 1968.
The author thanks Professor J. J. Levin for discussions of this work. This research was begun while the author held an NSF Graduate Fellowship. The author also wishes to acknowledge partial support from NSF grant GP-9658.