COBORDISM AND BUNDLES OVER SPHERES

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1. INTRODUCTION

P. E. Conner and E. E. Floyd [3] have determined when a (smooth, closed) manifold is cobordant modulo 2 to a bundle over the 1-sphere S^1. If w_i is the ith Stiefel-Whitney class of M^n and σ_n is the modulo-2 fundamental class, then M^n is cobordant to a bundle over S^1 if and only if the Stiefel-Whitney number $\langle w_n, \sigma_n \rangle$ vanishes. We study the analogous question for bundles over S^2. Let

$$C(M^n) = \begin{cases} \langle w_n, \sigma_n \rangle & \text{(n even)}, \\ \langle w_2 \cdot w_{n-2}, \sigma_n \rangle & \text{(n odd)}. \end{cases}$$

THEOREM 1. *If* M^n *is cobordant to a bundle over* S^2, *then* $C(M^n) = 0$. *If* $C(M^n) = 0$, *then* M^n *is cobordant to a connected bundle over* S^2 *with group* $U(1)$.

This answers in the negative the question of Conner [2, end of paper] whether the generator of the oriented cobordism group $\Omega_5 \cong Z_2$ can be represented by a bundle over S^2. However, we do not know the complete oriented analogue of Theorem 1.

According to Corollary 6.2 of [3], the square of a bundle over S^1 is cobordant to a bundle over S^2. We prove a generalization and an analogue of this result.

COROLLARY 1. *The product of two bundles over* S^1 *is cobordant to a bundle over* S^2.

THEOREM 2. *The square of a bundle over* S^2 *is cobordant to a bundle over* S^4.

In Section 2, we derive some necessary conditions for a manifold to be a bundle over S^k $(k > 0)$ by applying the theorem of E. H. Brown and F. P. Peterson [1, Section 1] on relations among characteristic classes to the fiber of the bundle. In Section 3, we construct some bundles over S^2, S^4, and S^8; and in Section 4, we show that we have enough bundles over S^2 to generate the kernel of the character $C: \mathfrak{N}_* \to Z_2$, where $\mathfrak{N}_*$ denotes the unoriented cobordism ring. In Section 5, we show how to make our bundles connected, and in Section 6 we prove Theorem 2.

An optimistic conjecture is that the necessary conditions derived from Proposition 2.1 are sufficient for a manifold to be cobordant to a bundle over S^k for $k = 1, 2, 4, 8$. (They are, for $k = 1, 2$.) For other values of k, see Theorem 8.1 of [3].

2. NECESSARY CONDITIONS

Let $k > 0$ and $n > k$, and let M^n be a bundle over S^k with projection $p: M^n \to S^k$ and fiber F. The tangent bundle of M^n decomposes as a direct sum

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