ON THE INTEGRAL COHOMOLOGY GROUPS OF THE CLASSIFYING SPACE FOR BSO

Robert R. Clough

1. INTRODUCTION

E. Thomas has shown [5] that all torsion in $H^*(B\,Spin; Z)$ is of order 2; a result corresponding to the long-known result for BSO. In the study of spaces realizing the image of the stable J-homomorphism [2], the cohomology of the classifying space BBSO for BSO regarded as an H-space is of interest. J. D. Stasheff [3] has found torsion of order 2^n for each n in $H^*(BBSO; Z)$. In this paper, we shall present results for exterior algebras analogous to the results of Thomas for polynomial algebras in [4] and [5], in order to show that Stasheff has found essentially all of the higher torsion in $H^*(BBSO; Z)$.

If X is a graded set and $\lambda: X \to N = \{0, 1, 2, \cdots\}$ is a function, then $A[\lambda]$ is the graded, abelian group generated by the elements of X subject to the conditions $\lambda(x)x = 0$.

THEOREM 1. $H^*(BBSO; Z)$ *and* $(E \otimes A[\lambda]) \oplus T$ *are isomorphic as groups, where*

(1) E *is the graded, anticommutative,* Z*-exterior algebra on one class* P_{4i+1} *of each degree* $4i + 1 \geq 5$;

(2) $\lambda: \{a_{8n} \mid n \geq 1\} \to N$ *is defined by*

$$\lambda a_{8n} = \text{the greatest integer } 2^k \text{ that divides } 4n,$$

and degree $(a_{8n}) = 8n$;

(3) $2T = 0$.

2. PROOF OF THEOREM 1

Suppose X is a space each of whose integral cohomology groups is finitely generated. Let

$$\rho: H^*(\ ; Z) \to H^*(\ ; Z_2) \quad \text{and} \quad \rho': H^*(\ ; Z) \to H^*(\ ; Q)$$

be universal coefficient maps. Let $\{u_1, u_2, \cdots\} \subset H^*(X; Z)$, where the u_n are all of odd degree. We shall prove the following analogue of Theorem 4.2 in [5].

PROPOSITION 1. *Suppose the cohomology groups of* X *satisfy the following three conditions:*

(1) $H^*(X; Z)$ *has no odd torsion.*

(2) $H^*(X; Z_2) = Z_2\, E[w_1, w_2, \cdots; x_1, x_2, \cdots; y_1, y_2, \cdots;$

$$z_0, z_1, z_2, z_4, \cdots, z_{2^i}, \cdots],$$

Received July 29, 1968.