MULTIPLICATIONS ON PROJECTIVE SPACES

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In this paper, we consider the collection of multiplications on projective spaces. The results of [1] imply that the only projective spaces that admit a multiplication are the real projective spaces P^n, for $n = 1$, 3, and 7. We describe the collection of multiplications on P^3 as a group and determine the number of multiplications on P^7. C. M. Naylor [4] previously found the number of multiplications on P^3.

1. Let ϕ: $X \vee X \to X$ denote the "folding" map.

A multiplication on a space X is a map μ: $X \times X \to X$ such that $\mu \mid X \vee X = \phi$. Two multiplications on a space X are said to be homotopic if they are homotopic as maps relative to $X \vee X$.

M. Arkowitz and C. R. Curjel showed in [2] that if X is a finite CW-complex admitting a multiplication, then there exists a one-to-one correspondence between the set of homotopy classes of multiplications on X and the homotopy set $[X \wedge X, X]$. When X has a homotopy associative multiplication, $[X \wedge X, X]$ has a group structure.

LEMMA 1. *Let* M *be a smooth, connected manifold of dimension* n, *and let* M_0 *be* M *with an open disc removed. If there exists a smooth embedding* f: $M \to S^{m+n}$ *with trivial normal bundle, then* $S^m M \simeq S^m M_0 \vee S^{n+m}$.

Proof. Let N denote a closed tubular neighbourhood of the embedding, and let T denote the Thom complex of the normal bundle. T is $N/\partial N$ by definition, and it is homotopically equivalent to $S^m \vee S^m M$. If D is a small disc of dimension $m + n$ lying in the interior of N, the space T - D is homotopically equivalent to $S^m \vee S^m M_0$. The attaching map of D is homotopically trivial in S^{n+m} and therefore is also trivial in T. This proves that

$$S^m \vee S^m M \simeq S^m \vee S^m M_0 \vee S^{n+m},$$

and therefore we have that $S^m M \simeq S^m M_0 \vee S^{n+m}$.

I am extremely grateful to Dr. B. J. Sanderson for showing me this lemma. It enables us to avoid a rather long direct proof of the following statement (when $n = 6$).

COROLLARY 2. *The covering map* π: $S^n \to P^n$ *is stably trivial when* $n = 2$ *or* $n = 6$.

Proof. We prove the result for $n = 6$. The case $n = 2$ is similar (and easy to prove directly, anyway).

Embed P^7 in R^{15}. Since P^7 is parallelizable, the normal bundle is trivial. The space P^7 with an open disc removed is homotopically equivalent to P^6, and the attaching map for this disc is $S^8\pi$. It follows from the proof of the lemma that $S^8\pi \simeq 0$.

LEMMA 3. *If* K *is an* n-*dimensional complex, then* S: $[K, S^m] \to [SK, S^{m+1}]$ *is an isomorphism provided*

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