AN EXTENSION OF WEYL'S THEOREM TO A CLASS OF NOT NECESSARILY NORMAL OPERATORS

S. K. Berberian

1. INTRODUCTION

A bounded linear operator T on a Hilbert space is called a *Fredholm operator* if its null space N(T) is finite-dimensional and its range R(T) is a closed subspace of finite codimension (see [4, p. 91]); the *index* of a Fredholm operator is defined as

$$i(T) = \dim N(T) - \dim R(T)^{\perp} \qquad (= \dim N(T) - \dim N(T^*)).$$

Every invertible operator is trivially a Fredholm operator with index 0.

The *spectrum* $\sigma(T)$ of T is defined by the formula

$$\sigma(T) = \mathbf{C}\,\{\lambda\colon T - \lambda I \text{ is invertible}\}.$$

Analogously, the *Weyl spectrum* $\omega(T)$ of T is defined by the formula

(1) $$\omega(T) = \mathbf{C}\,\{\lambda\colon T - \lambda I \text{ is a Fredholm operator of index } 0\}.$$

Obviously, $\omega(T) \subset \sigma(T)$. The concept of Weyl spectrum is relevant only for infinite-dimensional spaces: $\omega(T) = \emptyset$ when the space is finite-dimensional, all operators being Fredholm operators of index 0. For technical reasons, it is convenient to allow finite-dimensional spaces. When the space is infinite-dimensional, $\omega(0) = \{0\}$.

The Weyl spectrum (also called the essential spectrum) occurs in the theory of perturbation by compact operators; it has the agreeable property of being invariant under such perturbation; that is, $\omega(T + K) = \omega(T)$ for all compact K (see [9]). We shall not use this property, but we remark that it implies that

$$\omega(T) = \bigcap \{\sigma(T + K)\colon K \text{ is compact}\}$$

(L. A. Coburn [3] used this formula to define Weyl spectrum), and that $\omega(T) = \{0\}$ when T is compact and the space is infinite-dimensional.

Modulo the above remarks, the Weyl spectrum tends to be large:

LEMMA 1. $\sigma(T) - \omega(T)$ *is either empty or consists of eigenvalues of finite multiplicity.*

Proof. If $\lambda \in \sigma(T) - \omega(T)$, then $S = T - \lambda I$ is singular and is a Fredholm operator of index 0. In particular, $\dim N(S) < \infty$, thus the problem is to show that $N(S) \neq \{0\}$. Assume to the contrary that S is injective; then $0 = i(S) = 0 - \dim R(S)^{\perp}$, and since R(S) is closed, this implies S is surjective. Thus S is bijective and therefore invertible [4, Problem 41], a contradiction.

Extending a classical result of H. Weyl for normal operators, Coburn [3] showed that if T is any hyponormal operator, then

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