ON MAPS WITH NORMAL STRUCTURE

Thomas J. Kyrouz

INTRODUCTION

In this note, we show that smooth imbeddings that are homotopic through smooth imbeddings have fiber-homotopy-equivalent normal sphere bundles, provided the codimension is at least 3.

The method is to consider maps having (generalized) tubular neighborhoods. Each such neighborhood gives rise to a corresponding normal fibering, which in the smooth case is fiber-homotopy equivalent to the normal sphere bundle. The invariance under homotopy is deduced from a uniqueness theorem for such neighborhoods. The situation for codimension 2 is discussed in the last section.

1. NORMAL STRUCTURES

Definition 1. Suppose K is a finite complex, M^n is a manifold, and f: K → M is a map. A T-*neighborhood* for f is a compact manifold $N^n \subset M^n$ such that $f(K) \subset \text{int}\, N$ and f: K → N is a homotopy equivalence. Two T-neighborhoods N and N' for f are said to be *equivalent* if there is a homotopy equivalence of pairs h: $(N, \partial N) \to (N', \partial N')$ such that f and hf are homotopic as maps from K to N'. An equivalence class $\mathcal{N}$ of T-neighborhoods for f is called a *normal structure* if each open neighborhood of f(K) contains a member of $\mathcal{N}$. The *formal codimension* of $\mathcal{N}$ is the least integer k such that $\pi_k(N, \partial N) \neq 0$ $(N \in \mathcal{N})$.

THEOREM 1. *A map* f: K → M *admits at most one normal structure of formal codimension greater than or equal to* 3.

Proof. Let $\mathcal{N}$ and $\mathcal{N}'$ be normal structures for f, of formal codimension at least 3. Let $N \in \mathcal{N}$, and choose $N' \in \mathcal{N}'$ so that $N' \subset \text{int}\, N$ and N - int N' = W is a manifold. We now show that W is an h-cobordism. Since the formal codimension is at least 3, we have isomorphisms $\pi_1 \partial N' \to \pi_1 N'$ and $\pi_1 \partial N \to \pi_1 N$. The theorem of Van Kampen applied to $N = N' \cup W$ shows that $\pi_1 \partial N' \to \pi_1 W$ is an isomorphism, and it follows easily that $\pi_1 \partial N \to \pi_1 W$ is an isomorphism. Passing to universal covering spaces, we see that we can obtain $(\tilde{W}, \partial\tilde{N}')$ from $(\tilde{N}, \tilde{N}')$ by excising the part over int $\tilde{N}'$. Since N' → N is a homotopy equivalence, $H_*(\tilde{N}, \tilde{N}') = 0$, and therefore $\partial\tilde{N}' \to \tilde{W}$ and $\partial N' \to W$ are homotopy equivalences. Poincaré duality gives $H_*(\tilde{W}, \partial\tilde{N}) = 0$, and it follows that $\partial N \to W$ is a homotopy equivalence. Now let r: $W \to \partial N'$ be a deformation retraction. Taking r and $1_{N'}$ together, we obtain h: N → N', which is an equivalence of T-neighborhoods.

Remark. W may fail to be an h-cobordism in codimension 2; take f: $S^1 \to S^1 \times D^2 = N$ to be a trefoil knot homotopic to the zero section, and N' to be a smooth tubular neighborhood of f. Notice that N and N' are equivalent as T-neighborhoods for f.

Received September 13, 1968.
This work was supported by N.S.F. grant GP-8615.