A CHARACTERIZATION OF UNIFORMLY CONTINUOUS UNITARY REPRESENTATIONS OF CONNECTED LOCALLY COMPACT GROUPS

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In this paper, we characterize the uniformly continuous unitary representations of connected, locally compact groups. Roughly stated, the main theorem says that a unitary representation of a connected, locally compact group is uniformly continuous if and only if its support (see J. Dixmier [1, Definition 18.1.7, p. 315]) is a nice bounded set.

In what follows, G denotes a connected, locally compact group whose topology satisfies the second axiom of countability. Let $H = \bigcap_\pi \operatorname{Ker} \pi$, where π ranges over the set of finite-dimensional unitary representations of G. Then H is a closed, normal subgroup of G. Hence, G/H is also a connected, locally compact group whose topology satisfies the second axiom of countability.

The notation used in this paper is that of Dixmier [1]. Specific notation and results from [1] will be recalled as the need arises. To avoid needless circumlocution, we abbreviate "strongly continuous unitary representation" to "unitary representation."

The main result is that $\pi(\cdot)$ is a uniformly continuous unitary representation of G if and only if $\pi(\cdot)$ is quasi-equivalent to a direct integral

$$\sum_{\ell=1}^{n} \oplus \int_{\hat{G}_\ell} \pi(\xi)(\cdot)\, d\mu_\ell(\xi),$$

where n is a positive integer depending on π, and where μ_ℓ is a Borel measure on $\hat{G}_\ell$ $(1 \le \ell \le n)$ with compact support. In the process of proving this, we characterize the compact subsets of $\hat{G}_m$ (m a positive integer). Each compact subset of $\hat{G}_m$ is the union of finitely many sets of the form $(\hat{\pi}, C)$. Here $\hat{\pi} \in \hat{G}_n$, C is a compact subset of $\hat{G}_1$ (the set of characters of G), and

$$(\hat{\pi}, C) = [\hat{\pi}\chi \mid \chi \in C].$$

We first prove the theorem for the connected group G/H, making essential use of the fact that G/H is the direct product of a compact group and a vector group. Then we show that $(\widehat{G/H})_n$ and $\hat{G}_n$ are homeomorphic in a natural manner. The main result then follows from this.

LEMMA 1. G/H *is the direct product of a connected compact group and a vector group.*

Proof. By construction, G/H has a separating collection of finite-dimensional unitary representations. The theorem now follows from a result of R. V. Kadison and I. M. Singer [2, Theorem 1, p. 420]. ∎

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