SMOOTH HOMOTOPY LENS SPACES

Peter Orlik

To R. L. Wilder, with warm appreciation

1. INTRODUCTION

In the following, all manifolds are assumed to be smooth (unless it is otherwise stated) and all actions are differentiable. We are interested in free actions of finite cyclic groups on homotopy spheres.

The existence of free involutions on homotopy spheres has been studied in some detail. In particular, F. Hirzebruch [2] and Orlik and C. P. Rourke [8] proved that every element of the group $\theta^{4k+3}(\partial\pi)$ $(k \geq 1)$ of homotopy spheres that bound π-manifolds admits free involutions. It follows from a result of E. Brieskorn [1] that the same is true of the (possibly) nontrivial element of $\theta^{4k+1}(\partial\pi)$. Since $\theta^{2k}(\partial\pi) = 0$, we again have a free Z_2-action.

If $m > 2$, then clearly only odd-dimensional spheres can admit free Z_m-actions. In Section 2, we define free actions of Z_m and fixed-point-free actions of $U(1)$ on Brieskorn spheres. We use these in Section 3 to prove that *for each prime* p, *every element of* $\theta^{2k+1}(\partial\pi)$ $(k > 1)$ *admits a free action of* Z_p. This contrasts with the fact that not every element of $\theta^{2k+1}(\partial\pi)$ admits a *free* circle action.

In Section 4 we compare our actions with those obtained by J. Milnor [4], by D. Montgomery and C. T. Yang [5], and by C. N. Lee [3], and we show that some are definitely distinct from those previously known. In Section 5, we describe the Brieskorn spheres as branched finite cyclic coverings of the standard sphere, branched along a Brieskorn variety of codimension 2. This is used in Section 6 to determine the *homotopy types* of the orbit spaces of the Z_m-actions of Section 2. In Section 7 we determine their *stable tangent bundle* and *characteristic classes,* and in Section 8 we consider some normal bundles. In the 3-dimensional analogue we obtain free actions on homology spheres. A complete classification of the arising homology lens spaces is given in Section 9.

I am indebted to F. Hirzebruch for pointing out Lemma 2 and showing how it completes the argument of Theorem 1, to F. Raymond for helpful suggestions, and to R. Lee for stimulating conversations.

2. ACTIONS ON BRIESKORN SPHERES

Recall the variety

$$V(a) = \{z\colon z \in \mathbb{C}^{n+1},\ z_0^{a_0} + \cdots + z_n^{a_n} = 0\}$$

considered by E. Brieskorn [1]. Here $a = \{a_0, \cdots, a_n\}$ is a set of integers $(a_j \geq 2$ for each j). V(a) has an isolated singularity at the origin. Its intersection with the unit sphere in $\mathbb{C}^{n+1}$,

---

Received April 1, 1969.
The author acknowledges support from the Wisconsin Alumni Foundation.