

SUBDIRECT DECOMPOSITIONS OF EXTENSION RINGS

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Let A and S be two rings. We say that E is an *extension* of A by S if A is an ideal of E and E/A is isomorphic to S . The purpose of this note is to give a simple counterexample to a theorem of C. W. Kohls [1], and to prove a similar result. The result is strong enough to obtain several of the corollaries in [1].

The result of Kohls [1, p. 401] states that if A is isomorphic to a subdirect sum of simple primitive rings $\{A_\alpha\}$ and if S is isomorphic to a subdirect sum of primitive rings $\{B_\beta\}$, then every extension E of A by S is isomorphic to a subdirect sum of the rings $\{A_\alpha\} \cup \{B_\beta\}$. The counterexample is as follows: Let E be the ring of all linear transformations of a vector space of countably infinite dimension into itself. Let A be the ideal consisting of all elements of E of finite rank. It is well known that the only ideals of E are the zero ideal A and E . Also A is a simple primitive ring. E/A is a simple ring with unity, and hence it is primitive. The result of Kohls would imply that E is isomorphic to a subdirect sum of A and E/A . This is clearly impossible, since E is subdirectly irreducible.

We recall that a ring is *semiprime* if it contains no nonzero nilpotent ideals.

THEOREM. *If A is isomorphic to a subdirect sum of semiprime rings with unity $\{A_\alpha\}$, and if S is isomorphic to a subdirect sum of rings $\{B_\beta\}$, then each extension E of A by S is isomorphic to a subdirect sum of the rings $\{A_\alpha\} \cup \{B_\beta\}$.*

Proof. For each α and each β , let

$$M_\alpha = \{a \in A: a_\alpha = 0\} \quad \text{and} \quad P_\beta = \{s \in S: s_\beta = 0\},$$

where a_α and s_β denote components in the given subdirect sums. Then $\bigcap_\alpha M_\alpha = 0$ and $\bigcap_\beta P_\beta = 0$. Also, A/M_α is isomorphic to A_α and S/P_β is isomorphic to B_β . Now there exist ideals P'_β of E containing A such that $P_\beta = P'_\beta/A$ and $\bigcap P'_\beta = A$. Let $M'_\alpha = \{x \in E: Ax \subseteq M_\alpha\}$. M'_α is clearly an ideal of E . Let b be in $A \cap M'_\alpha$. Then $Ab \subseteq M_\alpha$, which implies $A \langle b \rangle \subseteq M_\alpha$, where $\langle b \rangle$ denotes the right ideal of A generated by b . Therefore, $\langle b \rangle^2 \subseteq M_\alpha$. Now M_α is a semiprime ideal of A ; therefore, $\langle b \rangle \subseteq M_\alpha$. Hence b is in M_α . Clearly, $M_\alpha \subseteq A \cap M'_\alpha$. Hence $M_\alpha = A \cap M'_\alpha$. Let $a + M_\alpha$ be the identity of A/M_α . If x is in E and y is in A , then

$$y(x - ax) = yx - yax \equiv yx - yx \pmod{M_\alpha},$$

since y is in A . Since the last quantity is zero, $y(x - ax)$ is in M_α . Hence, $x - ax$ is in M'_α for all x in E .

This implies that $E = M'_\alpha + A$; for if x is in E , then $x = (x - ax) + ax$. Therefore,

$$E/M'_\alpha = (M'_\alpha + A)/M'_\alpha \cong A/(M'_\alpha \cap A) = A/M_\alpha \cong A_\alpha.$$

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