# THE COMPACTNESS OF THE SET OF ARC CLUSTER SETS

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Let f be a continuous, complex-valued function defined in the unit disk D, let C be the unit circle, and let W be the Riemann sphere. For each point $p \in C$, let $\mathfrak{T}(p)$ be the set of all Jordan arcs contained in $D \cup \{p\}$ and having one endpoint at p. For each $t \in \mathfrak{T}(p)$, define the *cluster set of* f *at* p *relative to the arc* t by

$$C_t(f, p) = \bigcap_{r>0} \overline{f(t \cap \{z: |z - p| < r\})}.$$

By a continuum we shall mean a closed, connected, nonempty subset of W. We remark that under our definition, a set with exactly one element is a continuum, and that for each continuous function f and each $t \in \mathfrak{T}(p)$, the cluster set $C_t(f, p)$ is a continuum.

If A and B are two nonempty closed subsets of W, define

$$M(A, B) = \max(\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)),$$

where $d(w_1, w_2)$ is the chordal distance between $w_1$ and $w_2$. The distance M(A, B) is a metric on the set of all nonempty closed subsets of W. If we define

$$\mathfrak{C}_f(p) = \{C_t(f, p): t \in \mathfrak{T}(p)\},$$

that is, if $\mathfrak{C}_f(p)$ is the set whose elements are the sets $C_t(f, p)$, then the metric M topologizes the set $\mathfrak{C}_f(p)$ with what we shall call the M-topology. The purpose of this paper is to investigate conditions under which $\mathfrak{C}_f(p)$ is compact in the M-topology.

By an *ambiguous point* p *for the function* f we mean a point $p \in C$ for which there exist two arcs $t_1$ and $t_2$ in $\mathfrak{T}(p)$ such that $C_{t_1}(f, p) \cap C_{t_2}(f, p) = \emptyset$. Our main result is the following theorem.

THEOREM 1. *Let* f *be a continuous function in* D, *and let* p *be a point of* C. *If* p *is not an ambiguous point for* f, *then* $\mathfrak{C}_f(p)$ *is a compact set in the* M-*topology.*

*Proof.* Suppose $\mathfrak{C}_f(p)$ is not a compact set in the M-topology. Then there exist a sequence of continua $\{K_n\}$ and a continuum K such that $K_n \in \mathfrak{C}_f(p)$ for each positive integer n, and such that $K \notin \mathfrak{C}_f(p)$ and $M(K_n, K) \to 0$. For each positive integer n, let

$$H_n = \{z \in D: d(f(z), K_n) < 1/n \text{ and } |z - p| < 1/n\}.$$

Since $K_n \in \mathfrak{C}_f(p)$, there exist a component $G_n$ of $H_n$ and an arc $t_n \in \mathfrak{T}(p)$ such that $C_{t_n}(f, p) = K_n$ and $t_n \subset G_n \cup \{p\}$.

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