ON THE SOLUTION OF THE RIEMANN PROBLEM WITH GENERAL STEP DATA FOR AN EXTENDED CLASS OF HYPERBOLIC SYSTEMS

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1. The Riemann problem for a quasi-linear hyperbolic system of equations is a specific initial-value problem, namely

$$U_t + F(U)_x = 0,$$

$$U(0, x) = \begin{cases} U_\ell & (x < 0), \\ U_r & (x > 0). \end{cases}$$

Here F is a vector-valued function of $U = U(t, x) \in E^n$ $(n \geq 2,\ -\infty < x < \infty,\ t \geq 0)$, and U_ℓ and U_r are constant vectors. The assumption that the system is hyperbolic means that the matrix $dF(U)$ has real and distinct eigenvalues for all values of the argument U in question. By a solution of this problem we mean a function $U = U(x/t)$ consisting of $n + 1$ constant states separated by shock and rarefaction waves, satisfying the Rankine-Hugoniot condition across shocks, and satisfying the equation in the usual sense at all other points (see [5] for the definitions of these concepts). Such a solution is necessarily a weak solution in the sense of the theory of distributions.

In 1957, P. Lax [5] solved the Riemann problem for the case where U_ℓ and U_r are sufficiently close. In this paper, we shall allow U_ℓ and U_r to be arbitrary constant vectors, but we shall restrict ourselves to the case $n = 2$ for the same systems that were studied in [2]. For these systems we shall show that at each point U_ℓ in the plane there originate four smooth curves that divide the plane into four unbounded regions, with the property that if U_r lies in three of these regions, then the Riemann problem can be solved without any additional assumptions. We shall then show by an example (due essentially to J. L. Johnson [1], who also considered certain special cases of our results) that our assumptions are not strong enough to solve all Riemann problems if U_r lies in the fourth region. However, by putting additional assumptions on F, we can guarantee that all Riemann problems are solvable. Also, by generalizing the notion of solution, we are able to solve all Riemann problems without any additional restriction on F.

2. In the case $n = 2$, we can write our system in the form

$$u_t + f(u, v)_x = 0, \qquad v_t + g(u, v)_x = 0, \tag{1}$$

with initial data

$$(u(0, x), v(0, x)) = \begin{cases} U_\ell = (u_\ell, v_\ell) & (x < 0), \\ U_r = (u_r, v_r) & (x > 0). \end{cases} \tag{2}$$

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