ON PERIODICALLY PERTURBED CONSERVATIVE SYSTEMS

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1. INTRODUCTION

The main result of this paper concerns the differential equation

$$\frac{d^2 x}{dt^2} + \text{grad } G(x) = p(t), \tag{1.1}$$

where $p \in C(R, R^n)$, p is 2π-periodic, and $G \in C^2(R^n, R)$. The equation (1.1) can be interpreted physically as the Newtonian equations of motion of a mechanical system subject to conservative internal forces and periodic external forces.

Specifically, we show that if there exist an integer N and numbers μ_N and μ_{N+1} such that

$$N^2 < \mu_N \leq \mu_{N+1} < (N+1)^2 \tag{1.2}$$

and

$$\mu_N I_n \leq \left(\frac{\partial^2 G(a)}{\partial x_i \partial x_j} \right) \leq \mu_{N+1} I_n \tag{1.3}$$

for all $a \in R^n$, where I_n is the $n \times n$ identity matrix, then (1.1) has a 2π-periodic solution. This extends results of D. E. Leach [3] and W. S. Loud [4] in the one-dimensional case. Leach and Loud use polar coordinates in the plane, and their method is not applicable to higher dimensions, but they are able to establish uniqueness, which we do not consider here.

In the final section we are able to show both existence and uniqueness for a two-point boundary-value problem for (1.1). Both results depend on a preliminary lemma concerning Hammerstein operators; the lemma is a mild extension of a well known result of C. L. Dolph [1]. In the periodic case we also need an extension of Brouwer's fixed point theorem.

2. A PRELIMINARY LEMMA

Let H be a real Hilbert space, and let K: H → H be a completely continuous, linear symmetric, positive semidefinite operator. Let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

denote those values of λ for which the null space of $\lambda K - I$ (I denotes the identity operator on H) has positive dimension, and let the number of times each λ_n occurs

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