PRODUCTS OF SELF-ADJOINT OPERATORS

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Introduction. The purpose of this paper is to present some partial results concerning the problem of characterizing the bounded linear operators T on a Hilbert space H that admit a factorization as a product of two self-adjoint operators. We conjecture that an invertible operator T has this property if and only if T is similar to its adjoint. The main results are (a) a proof of the conjecture under the restriction that $\dim H < \infty$, and (b) a characterization of the operators that are unitarily equivalent to their adjoints. We also establish other sufficient conditions under which the conjecture is true.

1. We begin by considering the finite-dimensional case. Theorem 1 gives a reasonably good characterization of the product of two self-adjoint operators.

THEOREM 1. *If* H *is a finite-dimensional Hilbert space, then the following are equivalent conditions for an operator* T *on* H.

(1) T *is a product of two self-adjoint operators.*

(2) T *is a product of two self-adjoint operators, one of which is invertible.*

(3) *There exists an invertible self-adjoint operator* A *such that* TA *is self-adjoint.*

(4) *There exists an invertible self-adjoint operator* A *such that* $A^{-1}TA = T^*$.

(5) *There exists a basis of* H *with respect to which the matrix of* T *is real.*

(6) T *is similar to* T^*.

Proof. Carlson [1] proved the equivalence of the first five conditions. For the sake of completeness, we include here a substantial simplification of his arguments.

The implications (2) ⇒ (4) ⇒ (3) ⇒ (2) ⇒ (1) are clear, and therefore it suffices to prove (1) ⇒ (2) and the chain (5) ⇒ (6) ⇒ (2) ⇒ (6) ⇒ (5).

(5) ⇒ (6). Suppose that T has real matrix (a_{ij}) relative to the basis $\{e_i\}$. Choose an orthonormal basis $\{f_i\}$, and define an invertible operator S so that $Sf_i = e_i$. Then $S^{-1}TS$ has matrix (a_{ij}) relative to the orthonormal basis $\{f_i\}$. Since any matrix is similar to its transpose, it follows that $S^{-1}TS$ is similar to $(S^{-1}TS)^t = (S^{-1}TS)^*$. This implies that T is similar to T^*.

(6) ⇒ (2). Assume that $TS = ST^*$ for some invertible operator S. Taking adjoints, one sees easily that

$$T(e^{i\theta}S + e^{-i\theta}S^*) = (e^{i\theta}S + e^{-i\theta}S^*)T^*$$

for each real θ. Now the operator

$$A_\theta = e^{i\theta}S + e^{-i\theta}S^* = (SS^{*-1} + e^{-2i\theta})e^{i\theta}S^*$$

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