APPROXIMATION OF W_p-CONTINUITY SETS BY p-SIDON SETS

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1. INTRODUCTION

For two complex-valued functions x and y defined on the integers Z, we define $x*y$ by

$$x*y(n) = \sum x(n-k)y(k) \quad (k \in Z),$$

provided the series converges for each $n \in Z$. We let ℓ_p denote the Banach space of complex-valued functions x on Z such that the norm

$$\|x\|_{\ell_p} = \left(\sum |x(n)|^p\right)^{1/p}$$

is finite $\left(\sum \text{ will always indicate summation over } Z\right)$. Corresponding to each function a in the space J of the complex-valued functions on Z with finite support, we have a trigonometric polynomial

$$f(\theta) = \sum a(n)e^{in\theta}.$$

We let $\|f\|_{W_p}$ and $\|a\|_{W_p}$ denote the norm of the operator $x \to a*x$ on ℓ_p. We are interested in the algebra W_p, which we now define as follows: a complex-valued function f on the circle G (reals modulo 2π) is in W_p if and only if there exists a sequence $\{f_n\}$ $(n = 1, 2, \cdots)$ of trigonometric polynomials that converges uniformly to f and also satisfies the condition

$$\|f_n - f_m\|_{W_p} \to 0 \quad \text{as } m, n \to \infty.$$

Let $\|f\|_{W_p}$ denote $\lim_{m\to\infty} \|f_n\|_{W_p}$. One can show that with this norm and the pointwise operations, W_p is a Banach algebra with the circle as its maximal ideal space; however, for our purposes we only need to know that $\|f\|_{W_p}$ dominates the supremum norm of f. A short proof of this follows Lemma 2.3. It turns out that W_p is isomorphic to a closed subalgebra of multipliers (bounded operators that commute with all translation operators) on $L_p(Z) = \ell_p$; in fact, this seems to be the natural way to show that W_p is a Banach algebra. There is an extensive literature on multipliers for L_p-spaces over locally compact groups.

Our purpose is to obtain some results concerning W_p-continuity sets, which are defined in Section 4. It is easy to see that the definition is equivalent to saying that a compact subset E of G is a W_p-continuity set if $W_p | E = C(E)$, that is, if every continuous function on E is the restriction to E of some element in W_p. Since W_1 is isomorphic to $L_1(Z)$, the W_1-continuity sets are precisely the Helson sets. We shall be interested in the cases $1 < p < 2$ $(W_2 = C(G))$.

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