A NONSTANDARD APPROACH TO LINEAR FUNCTIONS

Lawrence D. Kugler

It is a well-known result, due to Cauchy, that all of the continuous solutions of the functional equation

$$f(x + y) = f(x) + f(y), \tag{1}$$

where f is required to be a real-valued function of a real variable, are given by $f(x) = mx$. In 1905, Hamel discovered discontinuous solutions of (1). His construction depended on the existence of a basis for the vector space R of real numbers over the rational field [2]. Further papers on the subject have dealt mainly with the problem of finding conditions on an additive function that guarantee its continuity. For example, one such condition, due to G. Darboux, is that the function be bounded on some interval [1, p. 109]. A nonstandard discussion of this condition may be found in [4, pp. 80-83].

The dichotomy between the continuous and discontinuous solutions of (1) is striking, especially in view of the simplicity of the equation. In the present paper, we use the methods of nonstandard analysis to investigate the question whether all solutions of (1) can be described in terms of linear functions of the form mx, where m may be either infinite or finite.

The problem of finding solutions to (1) is closely related to the problem of finding all the characters of an additive subgroup S of R, that is, all the complex-valued functions χ on S such that $|\chi(x)| = 1$ for all $x \in S$ and

$$\chi(x + y) = \chi(x)\chi(y) \tag{2}$$

for all $x, y \in S$.

The solutions of this character problem are of two kinds, continuous and discontinuous, all of the former having the form $\chi(x) = e^{imx}$ ($m \in R$). The following version of Kronecker's approximation theorem (see [3, p. 431]) gives important information about the discontinuous solutions of (2).

THEOREM. *If* S *is a subgroup of* R *and* χ *is a character of* S, *then for every* $\varepsilon > 0$ *and every finite subset* $\{x_1, \cdots, x_n\}$ *of* S, *there exists a continuous character* $\chi_0(x) = e^{imx}$ *such that*

$$|\chi(x_j) - \chi_0(x_j)| < \varepsilon \quad (j = 1, 2, \cdots, n).$$

It follows from the approximation theorem that for each character χ of a subgroup S of R, the relation $T_1(x, \varepsilon, m)$ defined by the inequality $|\chi(x) - e^{imx}| < \varepsilon$ is concurrent in the sense of A. Robinson [5, p. 31]. Hence there exists an element m of an enlargement *R of R such that $\chi(x) \simeq e^{imx}$ for all $x \in S$. Thus every character χ of S, continuous or not, is of the form $\chi(x) = {}^0(e^{imx})$ for some $m \in {}^*R$.

Received September 5, 1968.
This work was supported by National Science Foundation Grant No. GP-6955.