ON THE CONTINUITY OF A CLASS OF UNITARY REPRESENTATIONS

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Let $\{U_n\}$ be a sequence of unitary operators satisfying the conditions $U_{n+1}^2 = U_n$ for $n = 1, 2, \cdots$. Let E^n be the spectral measure on $[-\pi, \pi)$ associated with U_n. In general, E^{n+1} is obtained from an orthogonal splitting of E^n (see the remark at the end of this note). Let D be the group of dyadic rationals, topologized as a subset of the reals. The U_n give rise to the representation V_r of D defined by $V_{m/2^n} = U_n^m$.

In this note, we study the relation between the measures E^n and the continuity of V_r. If for example $E^n(X) = E^{n+1}(X/2)$ for all n and all Borel sets X, then V_r is continuous in the uniform operator topology. If $U_n = \lambda_n I$ and the numbers λ_n satisfy the conditions $\lambda_{n+1}^2 = \lambda_n$ and $\lambda_1 = 1$, and if $\{\lambda_n\}$ has no limit, then the resulting V_r is continuous only on the zero vector. In general, the closed subspace C of vectors x for which $V_r x$ is a strongly continuous function of r is a proper subspace. The theorem we prove below tells how to recapture C from the E^n. The theorem bears out the feeling that the way to get continuity is to use principal square roots, at least asymptotically.

Throughout this note, B denotes the class of Borel sets on the line, and all limits involving projections are in the strong operator topology.

THEOREM. *If* P *is the orthogonal projection on* C *and* I *is any interval having* 0 *in its interior, then*

$$P = \lim_{m\to\infty} \lim_{n\to\infty} E^n(m\,2^{-n} I).$$

LEMMA. *If* $X \in B$ *and* $X \subset [-\pi, \pi)$, *then* $E^{n+1}(X/2) \leq E^n(X)$.

Proof. By the regularity of the spectral measures [1, p. 63], it suffices to prove the lemma for the case where X is an interval. Pick a sequence of "polynomials" p_m (we allow both positive and negative powers) such that $p_m(e^{i\lambda})$ converges boundedly to the characteristic function $\psi(\lambda)$ of X. Then $p_m(U_n) \to E^n(X)$ strongly. But

$$p_m(U_n) = p_m(U_{n+1}^2) = \int_{-\pi}^{\pi} p_m(e^{2i\lambda})\, dE^{n+1}(\lambda).$$

Let $X_0 = \{\lambda \in [-\pi, \pi): \lambda \in X/2 \pmod{\pi}\}$. Then $p_m(e^{2i\lambda})$ converges to the characteristic function of X_0 for $\lambda \in [-\pi, \pi)$, and therefore $p_m(U_n) = p_m(U_{n+1}^2)$ converges to $E^{n+1}(X_0)$ strongly. Hence $E^{n+1}(X/2) \leq E^{n+1}(X_0) = E^n(X)$.

Proof of the theorem. Define $F^n(X) = E^n(X/2^n)$ for each $X \in B$. Then, if $m \geq n$ and $X \subset [-2^n\pi, 2^n\pi)$, it follows from the lemma that

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