A STABILITY THEOREM FOR FRAMES IN HILBERT SPACE

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The point of departure for this note is the well-known theorem [1, p. 4] stating that if $\{x_i\}_1^\infty$ and $\{y_i\}_1^\infty$ are frames (that is, orthonormal sequences) in a Hilbert space H and if $\sum \|x_i - y_i\|^2 < \infty$, then the completeness of $\{x_i\}$ implies that of $\{y_i\}$ (for a generalization, see [2]). We may regard this theorem as a global extension of the trivial local stability theorem in which the symbol ∞ in the inequality is replaced with 1. In this note, we prove that any class of frames, stable under a small perturbation of a certain type, is automatically stable under a more radical perturbation.

1. SOME DEFINITIONS AND THE THEOREM

H will be a complex Hilbert space of arbitrary dimension, and A will be a fixed set such that $\operatorname{card}(A) \leq \dim(H)$. An A-frame is an H-valued function on A whose image is an orthonormal set of H.

Let R^+ be the nonnegative real numbers. Let ϕ be a function defined on $\prod_A R^+$ into $R^+ \cup \{\infty\}$ such that

(1) if $\xi_a \leq K\eta_a$ for all $a \in A$ and if $\phi\{\eta_a\} < \infty$, then $\phi\{\xi_a\} < \infty$,

(2) if for each $a \in A$, f_a is a continuous function from a topological space X into R^+, if $f_a(x) \leq \xi_a$ for all $a \in A$ and $x \in X$, and if $\phi\{\xi_a\} < \infty$, then $\phi\{f_a\}$ is continuous.

THEOREM. *Let* C *be a class of* A*-frames. Suppose there exists a* $\delta > 0$ *such that if* $\{x_a\} \in C$ *and* $\{y_a\}$ *is an* A*-frame with* $\phi\{\|x_a - y_a\|\} < \delta$, *then* $\{y_a\} \in C$. *Suppose further that* $\{x_a\} \in C$ *and that* $\{z_a\}$ *is an* A*-frame with* $\phi\{\|z_a - x_a\|\} < \infty$. *Then* $\{z_a\} \in C$.

Assumption (2), which is a generalized M-test, assures us that the perturbations defined by ϕ are small. We can obtain the prototypical theorem mentioned in the introductory paragraph by taking A to be the positive integers, $\phi\{\xi_j\} = \sum \xi_j^2$, $\delta = 1$, and C the class of complete frames. The theorem is an almost immediate consequence of the following lemma.

LEMMA. *Let* $\{x_a\}$ *and* $\{y_a\}$ *be* A*-frames. Then for each* t *with* $0 \leq t \leq 1$, *there exists an* A*-frame* $\{x_a(t)\}$ *satisfying the conditions*

(i) $x_a(0) = x_a$, $x_a(1) = y_a$,

(ii) $x_a(t)$ *is a strongly continuous function of* t,

(iii) $\|x_a(t) - x_a\| \leq 2\|y_a - x_a\|$ *for all* $a \in A$.

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