A CHARACTERISTIC PROPERTY OF THE EUCLIDEAN PLANE

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1. The euclidean plane $\mathbb{R}^2$ has the property that for each family of circles, the length of each circle is a linear function of the radius. It is less obvious that this *linear-growth* condition is satisfied by all curves in the plane, in the following sense: Let γ be a C^∞-curve of finite length and without self-intersection. For each $p \in \gamma$, let σ_p be a line segment lying on a chosen side of γ, and perpendicular to γ. The points on $\sigma(p)$ $(p \in \gamma)$ whose distance from γ is s form a curve γ_s, which is also of class C^∞ for all sufficiently small s. Let $\ell(\gamma_s)$ denote the length of γ_s. Then $\ell(\gamma_s) - \ell(\gamma)$ is a linear function of s (see Section 3).

This suggests a natural problem concerning Riemannian manifolds of arbitrary dimension. Let $\overline{M}$ be a Riemannian manifold, and let M be a compact, orientable submanifold of codimension one (*possibly with boundary*). In short, let M be a compact hypersurface in $\overline{M}$. For sufficiently small s, let M_s be the set of points lying on geodesics normal to M (and on a fixed side of M) at distance s from M. Let $\mathscr{A}$ denote the area (or volume) function, and consider the real-valued function $\phi(s) = \mathscr{A}(M_s) - \mathscr{A}(M)$. How does ϕ grow as a function of s?

In general, ϕ is quite arbitrary. Thus if $\overline{M} = \mathbb{R}^d$ and M is a sphere, then $\phi(s) = cs^{d-1}$, where c is some constant. On the other hand, if $\overline{M} = S^d$ (the d-dimensional sphere) and M is the great sphere in S^d, then the growth of ϕ is dominated by a linear function (see the proposition of Section 2). My problem is to determine all Riemannian manifolds $\overline{M}$ for which ϕ is a *linear* function for every compact (orientable) hypersurface M. A Riemannian manifold $\overline{M}$ with this property is said to have the *linear-growth property*. The solution of the problem is exceedingly simple:

THEOREM. *A Riemannian manifold has the linear-growth property if and only if it is locally the euclidean plane.*

The referee has kindly brought my attention to the fact that what I called M_s in the second paragraph above is usually referred to in the classical literature as a "parallel-body." The behavior of the growth of the volume of M_s when the ambient space $\overline{M}$ is the euclidean space was first considered by J. Steiner in 1840. For further details, the reader is referred to H. Hadwiger [3, p. 213]. The paper [4] by H. Weyl is also relevant here.

2. The proof of our theorem depends on the computation of the second variation of area. Since the situation is different from the standard one in the theory of minimal surfaces, we shall give complete details.

We deal separately with two cases. In the first case, $\dim \overline{M} = 2$. Here M is a finite C^∞-curve, and it is evidently the diffeomorphic image of either $[0, 1]$ or $[0, 1)$. (We usually do not distinguish between the map of a submanifold and its image.) For each $m \in M$, let σ_m denote a geodesic lying on one side of M, emanating at m, and normal to M at m. For convenience, we shall require further that $\sigma_m: [0, 1] \to \overline{M}$ is such that $\sigma_m(0) = m$ and $\|\sigma'_m(0)\| = 1$, where σ'_m denotes

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