COMPACT, ACYCLIC SUBSETS OF THREE-MANIFOLDS

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1. INTRODUCTION

Let G be a nontrivial abelian group. If X is a compact absolute neighborhood retract (abbreviated: CANR), then X is said to be G-*acyclic* if it is connected and the homology groups $H_i(X; G)$ vanish for each $i > 0$. We shall be concerned with the cases G = Z (the additive group of integers) and $G = Z_2$ (the integers modulo two). We present here some theorems that we believe will frequently be useful in proving that a Z_2-acyclic CANR X embedded in a 3-manifold M^3 is a compact absolute retract (CAR). This turns out to be the case, for example, if M^3 is Euclidean 3-space E^3, and a question of Borsuk [3, p. 216] is thus answered in the affirmative. In fact, it follows from Corollary 4.1 that a Z_2-acyclic CANR X in M^3 is a CAR provided $H_1(M^3; Z)$ is a free abelian group, and provided that every Z-acyclic finite polyhedron in M^3 is simply connected.

A G-acyclic CANR X in M^3 actually possesses a property that we call *strongly* G-*acyclic* (see Section 3), and many of our proofs use this alternate hypothesis. This permits applications to other problems. A *compact decomposition* of M^3 is a decomposition whose elements consist of the components of a compact set $S \subset M^3$, plus the individual points of $M^3 - S$. Such a decomposition is upper-semicontinuous (see [8]). A corollary of Theorem 5 is that if G is a compact decomposition of the 3-sphere S^3 and the decomposition space S^3/G is a 3-manifold, then each element of G is cellular. In fact, it follows from our results and from those of R. J. Bean in [2] that an equivalence between S^3 and S^3/G can be demonstrated by means of a pseudo-isotopy.

Some of our results are valid with either of the coefficient groups Z or Z_2. In this case, terms such as Z_*-*acyclic* or Z_*-*homology* will be used, with the understanding that the reader may interpret Z_* consistently as either Z or Z_2 in a given proof or discussion.

We adopt the convention that manifolds are connected. A *closed* manifold is compact and without boundary. We use the terms "surface" and "closed 2-manifold" interchangeably. "Mapping" means "continuous mapping"; S^n denotes the n-sphere. If $f: X \times [0, 1] \to Y$ is a mapping and $t \in [0, 1]$, we let $f_t: X \to Y$ denote the mapping defined by $f_t(x) = f(x, t)$, and we say that $f_t: X \to Y$ is a *homotopy*. A *loop in* Y is a mapping $f: S^1 \to Y$. If loops f_0 and f_1 in Y are homotopic as mappings of S^1 into Y, we call them *freely homotopic*, as opposed to "base-point preserving" homotopic.

Finally, the algebraic topology used has a strongly geometric orientation. A good reference is [11]. For information on CANR's, see [3] or [5].

2. SIMPLE MOVES IN THREE-MANIFOLDS

Throughout this section, M^3 will denote an orientable, nonclosed, piecewise-linear 3-manifold, and Z^3 will denote a compact polyhedron in Int M^3 such that

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