ON 3-MANIFOLDS THAT COVER THEMSELVES

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1. Let M be a compact, connected 3-manifold. We say that M *covers itself* if there is a nontrivial covering projection p: $M \to M$. We classify all nonprime 3-manifolds with this property, and we show that certain prime 3-manifolds fiber over the circle in the sense of Stallings [12].

This work was suggested by Kwun [6], who considered the class of closed, connected, orientable 3-manifolds (without boundary) that double-cover themselves. Kwun succeeded in classifying all nonprime manifolds in this class, and he showed that under certain technical restrictions the prime manifolds fiber over the circle. His results are special cases of our Theorems 1 and 3.

2. Recall that a closed 3-manifold is *prime* if it is not the connected sum of two 3-manifolds each of which is different from S^3. A compact 3-manifold with connected boundary is *prime* if it is not the disk sum of two 3-manifolds, each different from the closed 3-cell.

Milnor [9] has shown that every closed, orientable 3-manifold M is homeomorphic to a sum $P_1 \# P_2 \# \cdots \# P_n$ of prime manifolds, where the summands P_i are uniquely determined up to order and homeomorphism. (J. L. Gross [1], [2] has obtained a result analogous to Milnor's in the case where M is an orientable 3-manifold with nonvoid, connected boundary.) Raymond [11] observed that Kneser [5] actually "proved," modulo the validity of Dehn's lemma, a unique decomposition theorem for closed 3-manifolds, orientable or not. Kneser's theorem states that every closed 3-manifold can be written uniquely in normal form as the sum of prime manifolds ("in normal form" means that the number of nonorientable handles N in the sum is minimal).

Milnor proved that, with the exception of S^3 and $S^1 \times S^2$, an orientable, closed 3-manifold is prime if and only if it is irreducible. Raymond also observed that Milnor's proof extends to the nonorientable case if N is excluded.

In the next section, we classify closed, nonprime 3-manifolds that cover themselves.

3. Let P_3 denote real, projective 3-space, and consider the manifold $P_3 \# P_3$ (orientation need not be specified, because P_3 admits an orientation-reversing homeomorphism). For every integer $k > 0$, $P_3 \# P_3$ admits a k-sheeted covering by itself. This is exceptional behavior for a nonprime manifold, as the following theorem shows.

THEOREM 1. *A closed, connected, nonprime 3-manifold* M *covers itself if and only if* $M \approx P_3 \# P_3$.

First we indicate the unique manner in which $P_3 \# P_3$ covers itself, and then we devote the remainder of this section to three lemmas that complete the proof of

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