PIECEWISE LINEAR INVOLUTIONS OF $S^1 \times S^2$

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1. INTRODUCTION

Let S^n denote the triangulated n-sphere. Starting with S^0, we define S^n inductively as the suspension of S^{n-1}. Let k_n denote the simplicial involution of S^n that leaves S^{n-1} pointwise fixed and interchanges the suspension vertices. Define two involutions h_1 and h_2 of $S^1 \times S^2$ by

$$h_1(x, y) = (k_1(x), y) \quad \text{and} \quad h_2(x, y) = (x, k_2(y)).$$

In this paper, we prove the following uniqueness results.

THEOREM 1. *Let* h *be a piecewise linear* (PL) *involution of* $S^1 \times S^2$, *with homogeneously* 2-*dimensional fixed point set* F. *If* F *is not connected, then* h *is* PL-*equivalent to* h_1.

THEOREM 2. *Let* h *be a* PL *involution of* $S^1 \times S^2$ *with* 2-*dimensional connected fixed point set* F *and orientable orbit space. Then* h *is* PL-*equivalent to* h_2.

There is a PL-involution of $S^1 \times S^2$ that has a Klein bottle as fixed point set and a nonorientable 2-disk bundle over S^1 as orbit space. Hence the orientability of the orbit space in Theorem 2 is essential. It is equivalent to the assumption that F separates $S^1 \times S^2$.

The following remark applies to both theorems. Since F is 2-dimensional, h has the property that near each point of F it maps one side of F to the other side. If this were not the case, one could produce a small invariant 2-sphere S near F such that $h \mid S$ has a 2-cell as fixed point set. But this is impossible. Hence, near each point of F (and therefore globally), h reverses the orientation. Throughout, we use the singular (or simplicial) homology, with integer coefficients unless it is otherwise specified.

2. PROOF OF THEOREM 1

PROPOSITION 1. F *separates* $S^1 \times S^2$.

Proof. Suppose $S^1 \times S^2$ - F is connected. Consider the homology sequence

$$0 \to H_3(S^1 \times S^2) \xrightarrow{j_*} H_3(S^1 \times S^2, F) \xrightarrow{\partial_*} H_2(F) \xrightarrow{i_*} H_2(S^1 \times S^2)$$

of $(S^1 \times S^2, F)$ over Z_2. Since $H_3(S^1 \times S^2, F) \simeq Z_2$, rank $H_2(F) \leq 1$. But rank $H_2(F)$ is the number of components of F. Since F is not connected, the result follows.

The following proposition implies that F is orientable.

PROPOSITION 2. $S^1 \times S^2$ - F *has exactly two components, and they are interchanged under* h.

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