A REMARK ON FREE MODULES

K. R. Mount

Suppose A is a commutative ring with identity, and $A[x]$ is a polynomial ring in one indeterminate with coefficients in A. Suppose F is a free module over $A[x]$ with a basis $e_0, \cdots, e_r$. Corresponding to each element P of F, denote by $\{P\}$ the submodule of F generated by P. Set $P = \sum P_u e_u$ with

$$P_u = \sum_{v=0}^{R} p(u, v) x^v \qquad (0 \leq u \leq r),$$

and assume that $Q_u = \sum_{v=0}^{d} q(u, v) x^v$ are $r + 1$ polynomials, each of degree d, the coefficients $q(u, v)$ being independent indeterminates. We shall denote by $E(P, n; d)$ the matrix of the system of linear equations (in the variables $q(u, v)$) obtained by equating to zero the coefficients of the x^j $(0 \leq j \leq n + d)$ in the expression $\sum (-1)^u P_u Q_u$. The matrix $E(P, rd; d - 1)$ is square. In this paper we prove the following proposition.

THEOREM. *If the determinant of* $E(P, rd; d - 1)$ *is a unit in* A *(here* rd *denotes the maximum of the degrees of the* P_u*) and if the* A*-module*

$$[Ae_0 + \cdots + Ae_r] / \left\{ \sum p(u, rd) e_u \right\}$$

is free, then the module $F / \left\{ \sum Q_u e_u \right\}$ *is free for each* $\sum Q_u e_u$ *such that* $\sum (-1)^u Q_u P_u = 1$. *Furthermore, if* A *is an integral domain, then* $F / \left\{ \sum P_u e_u \right\}$ *is free.*

We shall suppose throughout this paper that the rings discussed are commutative and have a unit. If $P = \sum P_u e_u$ is an element of F, we shall say that P has degree d if a polynomial of maximal degree occurring among the P_u has degree d. We shall refer to the matrix $E(P, n; d)$ as the d-th eliminate of P, and we shall suppose that the columns of $E(P, n; d)$ are indexed by the pairs (u, v) of integers with $0 \leq u \leq r$ and $0 \leq v \leq d$, while the rows are indexed by j $(0 \leq j \leq n + d)$. In case P has degree rd, the matrix $E(P, rd; d)$ has $(r + 1)(d + 1)$ columns and $(r + 1)d + 1$ rows; therefore, if A is a field, the dimension of the solution space of the equations $\sum P_u Q_u = 0$ (with $\deg Q_u \leq d$) is at least r.

Until we specify otherwise, we shall suppose that A is a field, and we shall denote by K an algebraically closed field of infinite degree of transcendence over A. We alter notation slightly to denote by F a free $K[x]$-module with basis $e_0, \cdots, e_r$. Denote by F_d the K-vector subspace of F consisting of elements of degree at most d. Denote by $G(F_d; r)$ the Grassmann space of r-dimensional subspaces of F_d. If V is a vector space, then $P(V)$ will denote the projective space consisting of the one-dimensional subspaces of V. We shall say that an element of $P(F_d)$ has degree t if a nonzero vector in that element has degree t.

Received March 8, 1968.
This work was supported in part by N.S.F. Grant No. GP-6853.