GROUPS OF ORDER AUTOMORPHISMS OF CERTAIN HOMOGENEOUS ORDERED SETS

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1. INTRODUCTION

Call a chain (linearly ordered set) *short* if it contains a countable unbounded subset, and *homogeneous* if all convex subsets without greatest or least elements are isomorphic. The purpose of this paper is to investigate the algebraic structure of the group $S(\Omega)$ of order automorphisms of a short homogeneous chain (abbreviated SHC) Ω.

In Section 2 we show that the group structure of $S(\Omega)$ determines, up to duality, the structure of $\overline{\Omega}$ (the conditional completion of Ω) and the lattice structure of $S(\Omega)$. We give a partial solution to the problem of finding all SHC's Ω with the same group $S(\Omega)$. Our solution includes the result $S(\mathscr{R}) \not\cong S(\mathscr{Q})$.

In Section 3 we calculate the automorphism groups of large subgroups of $S(\Omega)$. Our result includes the theorem of J. T. Lloyd [5] that if Ω is conditionally complete, then every automorphism of $S(\Omega)$ comes from conjugation by an order automorphism or antiautomorphism of Ω.

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Some notation: S^{Ω} is the full group of permutations of Ω; $L(\Omega)$ (respectively, $R(\Omega)$) is the subgroup of elements of $S(\Omega)$ whose support is bounded on the right (on the left); and $N(\Omega) = R(\Omega) \cap L(\Omega)$. For unexplained terminology, see [7] and [1].

We note that not every SHC is a subset of $\mathscr{R}$ (the set of real numbers). See, for example, [6].

2. GROUP STRUCTURE AND ORDER

The following is the fundamental tool of this paper.

THEOREM 1. *If* Ω *is short, and all of its open intervals are isomorphic, then* $L(\Omega)$, $R(\Omega)$, *and* $N(\Omega)$ *are the only proper normal subgroups of* $S(\Omega)$; *also,* $N(\Omega)$ *is the only proper normal subgroup of* $L(\Omega)$ *or* $R(\Omega)$, *and* $N(\Omega)$ *is algebraically simple.*

The difficult part of this, the simplicity of $N(\Omega)$, is due to G. Higman [2] (see also [7, p. 25]). The rest of Theorem 1 is a consequence of [3, Theorem 6]. A proof also appears in [5].

Note that if Ω is isomorphic to its order dual Ω^*, then $L(\Omega) \cong R(\Omega)$ and all four of the simple factors $S(\Omega)/L(\Omega)$, $S(\Omega)/R(\Omega)$, $R(\Omega)/N(\Omega)$, and $L(\Omega)/N(\Omega)$ are isomorphic. To complete the picture, we state without proof the following theorem.

THEOREM 2. *Under the hypothesis of Theorem* 1, $N(\Omega) \not\cong R(\Omega)/N(\Omega)$.

It is not to be hoped, even if Ω is an SHC, that the algebraic structure of $S(\Omega)$ will determine Ω; for example, if Γ is the set of irrational numbers and $\mathscr{Q}$ is the

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