MINIMUM CONVEXITY OF A HOLOMORPHIC FUNCTION, II

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1. STATEMENT OF RESULTS

Let $w = f(z)$ be a nonconstant holomorphic function defined in the open unit disc D. An *arc* at $e^{i\theta}$ is a curve $A \subset D$ such that $A \cup \{e^{i\theta}\}$ is a Jordan arc. Let A be an arc at $e^{i\theta}$, parametrized by $z(t)$ $(0 \leq t < 1)$, and define a family $\mathscr{H}_A$ as follows: $H \in \mathscr{H}_A$ if and only if H is a closed half-plane in the finite w-plane W and there exists a t_0 $(0 \leq t_0 < 1)$ such that $f(z(t)) \in H$ if $t_0 \leq t < 1$. If $\mathscr{H}_A = \emptyset$, set $F_A = W$; otherwise, set $F_A = \bigcap H$, where the intersection is taken over all $H \in \mathscr{H}_A$. Note that if $f(z)$ is bounded on A, then F_A is the convex hull of the cluster set of $f(z)$ on A at $e^{i\theta}$. Our first result is the following improvement of an earlier theorem [5, Theorem 1].

THEOREM 1. *For each* $e^{i\theta}$ *there exists an arc* α *at* $e^{i\theta}$ *such that* $F_\alpha \subset F_A$ *for each arc* A *at* $e^{i\theta}$.

If $F_\alpha = \emptyset$, $f(z)$ has the limit ∞ on α at $e^{i\theta}$, and, to be sure, in a rather special way. If $F_\alpha = \{a\}$, $f(z)$ has the limit a on α at $e^{i\theta}$.

Write $f(z) = u(z) + iv(z)$, where $u(z)$ and $v(z)$ are the real and imaginary parts of $f(z)$. A real or complex-valued function $g(z)$ defined in D is said to have the (finite or infinite) *asymptotic value* a *at* $e^{i\theta}$ provided there exists an arc at $e^{i\theta}$ on which $g(z)$ has the limit a at $e^{i\theta}$. For each $e^{i\theta}$, we shall be concerned with the validity of the following proposition:

$P(\theta)$: *If* $u(z)$ *and* $v(z)$ *have the finite asymptotic values* a *and* b, *respectively, at* $e^{i\theta}$, *then* $f(z)$ *has the asymptotic value* $a + bi$ *at* $e^{i\theta}$.

An immediate consequence of Theorem 1 is that for each $e^{i\theta}$, either $f(z)$ has the asymptotic value ∞ at $e^{i\theta}$, or $P(\theta)$ holds. This result contains a theorem of Gehring and Lohwater [4]. We shall prove a considerably stronger theorem, which we proceed to state.

Let $\mathscr{L}$ be the family of straight lines L in W such that $f(z) \notin L$ if $f'(z) = 0$. Note that for each $L \in \mathscr{L}$, $f(z)$ is one-to-one on each component of the preimage $f^{-1}(L)$. Let $\mathscr{L}^*$ be the family of all half-lines L^* in W,

$$L^* = \{w + \rho e^{i\phi}: \rho \geq 0\} \qquad (w \in W,\ 0 \leq \phi < 2\pi),$$

such that $L^* \subset L$ for some $L \in \mathscr{L}$. A subset $\mathfrak{S}$ of the unit circumference C is defined as follows: $e^{i\theta} \in \mathfrak{S}$ if and only if there exists an arc at $e^{i\theta}$ that $f(z)$ maps one-to-one onto some L^* in $\mathscr{L}^*$. Clearly, on such an arc $f(z)$ has the limit ∞ at $e^{i\theta}$.

THEOREM 2. *With the possible exception of at most countably many* $e^{i\theta}$, $P(\theta)$ *holds. Any exceptional* $e^{i\theta}$ *is in* $\mathfrak{S}$.

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