A COEFFICIENT PROBLEM FOR A CLASS OF UNIVALENT FUNCTIONS

T. J. Suffridge

1. INTRODUCTION

Let $E_r = \{z: |z| < r\}$, let $E_1 = E$, and let

$$S_r = \{f: f \in S \text{ and } f(E) \supset E_r\},$$

where S denotes the collection of functions $f(z) = z + a_2 z^2 + \cdots$ that are regular and univalent in E. Let S_r^* consist of the functions $f \in S_r$ for which $f(E)$ is starlike, and write $S_{1/4}^* = S^*$. Consider the following extremal problems.

Problem 1. Find $\max |a_2|$ for $f(z) = z + a_2 z^2 + \cdots \in S_r$.

Problem 2. Find $\max |a_2|$ for $f(z) = z + a_2 z^2 + \cdots \in S_r^*$.

It is clear that for $r \leq 1/4$ the Koebe function solves both problems, and that for $r = 1$ the function $f(z) = z$ solves both problems.

In this paper we solve Problem 2, and we make a conjecture concerning Problem 1. We also conjecture that for the class σ of biunivalent functions (see [4]), the coefficient a_2 satisfies the sharp inequality $|a_2| \leq 4/3$. We give an example of a function $f \in \sigma$ for which $|a_2| = 4/3$.

For $1/4 < r < 1$, an extremal domain for Problem 2 consists of the entire complex plane minus a set $\{z: |z| \geq r, \pi - \psi \leq \arg z \leq \pi + \psi\}$ $(0 < \psi < \pi)$. To be more specific, if we choose ϕ so that

$$(1) \qquad r = [(1 + \cos\phi)^{1+\cos\phi}(1 - \cos\phi)^{1-\cos\phi}]^{-1} \qquad (0 < \phi < \pi/2),$$

then $|a_2| = 2\cos^2\phi$ for the extremal function $f(z) = z + a_2 z^2 + \cdots$; and if we take $a_2 > 0$, the extremal domain is as described above, with $\psi = \pi(1 - \cos\phi)$.

It is interesting to note the relation between this solution and the solution to another extremal problem. Let $M > 1$, and consider the functions $g(z) = z + b_2 z^2 + \cdots$ in S with $|g(z)| < M$ $(z \in E)$. The function $g \in S$ determined by the differential equation

$$\frac{zg'(z)}{g(z)} = G(z) = \left(1 + \frac{4\left(1 - \frac{1}{M}\right)z}{(1 - z)^2}\right)^{-1/2} \qquad (G(0) = 1)$$

maximizes $|b_2|$ in this class of functions [5, p. 244, Exercise 4]. If we take $\cos^2\phi = 1 - \frac{1}{M}$ and r has the value in (1), then the function $f \in S$ satisfying the equation

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