A CONNECTION BETWEEN THE CESARI AND LERAY-SCHAUDER METHODS

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1. INTRODUCTION

A method that L. Cesari, J. K. Hale, and R. A. Gambill [2], [9], [12], [13], [14], [15] used to solve perturbation problems was generalized in 1963 and 1964 by Cesari [3], [5] so as to apply to strictly nonlinear problems. We shall call the method of [3] and [5] the *Cesari method*. Cesari, Hale, and H. W. Knobloch [8], [16], [17], [18] have since applied this method to boundary-value problems for ordinary and partial differential equations. S. Bancroft, J. K. Hale, and D. Sweet [1] and J. Locker [20] have extended the Cesari method in ways that we shall not consider here. Cesari [4], [6], [7] has proved the existence of periodic solutions of certain hyperbolic partial differential equations, solving his determining equation by use of the Tychonoff theorem in an infinite-dimensional space. In the present paper, however, we use only finite-dimensional methods (degree theory) to solve our determining equation.

By the term *Leray-Schauder method* we mean the method introduced in 1934 by J. Leray and J. Schauder [19].

Theorem 1 describes a theoretical link between the Cesari method and the Leray-Schauder method. Theorem 2 asserts the existence of a certain invariance property of an index (see the next section) associated with the Cesari method.

2. AN ABSTRACT DEFINITION OF THE CESARI INDEX

In defining the Cesari index below, we make several assumptions. Some of these assumptions made in [5]; the others are propositions proved in [5] as the results of assumptions of a more analytical nature. The reader may refer to Section 4 of the present paper for a comparison of the notation used in this paper with the notation in [5].

Let B be a Banach space, and let S be a finite-dimensional subspace of B. Let $P: B \to S$ be a projection, that is, let P be continuous and linear, with $P^2 = P$. Suppose that $\Gamma \subset B$, that $P\Gamma$ is compact, and that $(P^{-1}x) \cap \Gamma$ is closed for every x in $P\Gamma$. Let W be a continuous map from Γ into B. The Cesari method—after a suitable change in notation—gives sufficient conditions for W to have a fixed point in Γ.

Let I be the identity map in B, and let $T: \Gamma \to B$ be defined by $T = P + (I - P)W$. For each $x \in P\Gamma$, the restriction of T to $(P^{-1}x) \cap \Gamma$ is a map from $(P^{-1}x) \cap \Gamma$ into $P^{-1}x$. We shall assume that for each $x \in P\Gamma$ this restriction is a contraction from $(P^{-1}x) \cap \Gamma$ into itself, and we shall denote the resulting unique fixed point by $y(x)$ to indicate the dependence on x (see Remark 1 below). We shall assume that

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