LIFTING COMMUTING OPERATORS

R. G. Douglas, P. S. Muhly, and Carl Pearcy

1. INTRODUCTION

All Hilbert spaces considered in this paper are assumed to be complex, and all operators under consideration are assumed to be bounded and linear. The algebra of all (bounded, linear) operators on a Hilbert space $\mathscr{H}$ will be denoted by $\mathscr{L}(\mathscr{H})$, and if $S \in \mathscr{L}(\mathscr{H})$ and $\mathscr{M} \subset \mathscr{H}$ is an invariant subspace for S, then the operator in $\mathscr{L}(\mathscr{M})$ obtained by restricting S to $\mathscr{M}$ will be denoted by $S \mid \mathscr{M}$.

It is known [5] that if T is a contraction (that is, $\|T\| \leq 1$) acting on a Hilbert space $\mathscr{H}$, then there exists a unique minimal co-isometry V acting on a Hilbert space $\mathscr{K} \supset \mathscr{H}$ such that $T = V \mid \mathscr{H}$. Sarason showed in [4] that in case V^* is the unilateral shift of multiplicity one, then for every X in $\mathscr{L}(\mathscr{H})$ commuting with T, there exists a Y in $\mathscr{L}(\mathscr{K})$ such that $YV = VY$, $Y\mathscr{H} \subset \mathscr{H}$, $Y \mid \mathscr{H} = X$, and $\|Y\| = \|X\|$. Sarason's proof makes use of duality techniques and other properties of H^p-spaces. More recently, Sz.-Nagy and Foiaş generalized Sarason's result to the case that T is an arbitrary contraction [7, Theorem 2]. Their proof is based on the structure theory for contractions as set forth in [5].

In this paper we give an alternate and somewhat simpler proof of the above-mentioned lifting theorem of Sz.-Nagy and Foiaş. Our proof is matricial in character, and it employs an interesting generalization of a result of Douglas [1]. We then use [7, Theorem 2] to characterize the commutant of a contraction in terms of the commutant of its minimal strong unitary dilation.

Actually, in [7] Sz.-Nagy and Foiaş treat the more general case of two contractions T_1 and T_2 and an intertwining operator X satisfying the equation $XT_1 = T_2X$. However, we show in Section 5 that this more general result can easily be derived from the special case $T_1 = T_2$ by the use of a matricial device.

2. ON OPERATOR EQUATIONS

The basic tool for our attack on the lifting theorem is a generalization of the following lemma, which is contained in [1, Theorem 1].

LEMMA 2.1. *Suppose that $\mathscr{G}$, $\mathscr{H}$, and $\mathscr{K}$ are Hilbert spaces, that* A *is an operator mapping $\mathscr{G}$ into $\mathscr{K}$, and that* B *is an operator mapping $\mathscr{H}$ into $\mathscr{K}$. Then there exists a contraction* Z *mapping $\mathscr{G}$ into $\mathscr{H}$ and satisfying* $A = BZ$ *if and only if* $AA^* \leq BB^*$.

The following generalization is of interest in its own right.

THEOREM 1. *Let $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{K}$ be Hilbert spaces, and for* $0 \leq i \leq 2$ *let* A_i *be an operator mapping $\mathscr{H}_i$ into $\mathscr{K}$. Then there exist operators* Z_1 *and* Z_2 *that map $\mathscr{H}_0$ into $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and that satisfy the two conditions*

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