A RESULT ON GRAPH-COLORING

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The following problem was proposed by Erdös [4]:

Let $m > 1$ and $k \geq 1$ be integers, and let G be a graph with km nodes, each of the nodes having valence at least $(m - 1)k$. Does G contain k disjoint subgraphs each of which is a complete graph with m nodes?

In a number of cases an affirmative answer to Erdös' problem has been established (or follows from more general theorems). The author believes that the following list is complete:

(i) $m = 2$ (Dirac [3]);

(ii) $m = 3$ (Corrádi and Hajnal [2]);

(iii) $k \leq 3$ (Zelinka [5]).

It is the aim of the present note to enlarge the list by establishing an affirmative answer in the case $k = 4$.

Considering the graph complementary to the one of Erdös' problem (that is, reformulating Erdös' problem as a coloring problem) and making a mild generalization, one is led to the following conjecture (which reduces to Erdös' problem in case $n/k = m$ is an integer).

Conjecture. If a graph G has n nodes, and the valence of each node is less than k, then it is possible to color the nodes of G with k colors in such a manner that each color is assigned to at least $[n/k]$ nodes and to at most $[n/k] + 1$ nodes.

Zelinka's paper [5] adopts essentially the coloring point of view, and a trivial modification of his proof establishes the conjecture in case $k \leq 3$. The contribution of the present note is the proof of the conjecture in case $k = 4$.

THEOREM. *If* G *is a graph with* n *nodes and with maximal valence* 3, *it is possible to color the nodes of* G *by four colors in such a manner that each color is assigned either to* $[n/4]$ *or to* $[n/4] + 1$ *nodes.*

Proof. We use induction on the number of nodes of the graph G, noting that each node of G has valence at most 3. If every connected component of G has at most 3 nodes, the assertion is obviously true. If some connected component of G is the triod T (Figure 1) the assertion follows if we assign the four colors to the four nodes of T and use the inductive assumption to color the rest of G.

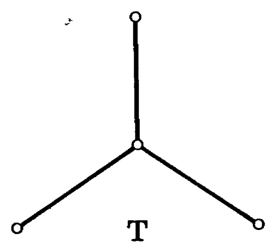

Figure 1

For the remaining part of the proof, we need the notion of a *nice 3-path*. We shall say that the nodes A_1, A_2, A_3, A_4 of a graph C with maximal valence 3 form a *nice 3-path* provided

(1) the nodes A_1, A_2, A_3, A_4 are all distinct;

(2) C contains the edges (A_1, A_2), (A_2, A_3), and (A_3, A_4) (and possibly additional edges between the A_i);

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