AVERAGES OVER A PAIR OF CONVEX SURFACES

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Let C_1 denote a convex curve in the plane and r the distance from a fixed point in the region bounded by C_1 to a variable point on C_1. In [3], Sachs obtained a result equivalent to the assertion that the average of r^2 with respect to arc length on C_1 is at least as large as the average of r^2 with respect to angle. We generalize this result in several directions.

First we replace r^2 with any increasing function of r; then we replace angular measure (which represents arc length on the unit circle) with arc length in a second convex curve; finally, we extend the result to convex surfaces in d-dimensional Euclidean space R^d.

The proof rests on the following theorem (see [4, page 146]): If A and B are convex bodies and $A \subseteq B$, then the measure of the surface of A is less than or equal to the measure of the surface of B. (If A is properly contained in B, the inequality is strict.)

In Section 2, we discuss nonconvex surfaces, in particular, the pedal curve of a convex curve.

1. AVERAGES OF A MONOTONIC FUNCTION OVER CONVEX SURFACES

If $X \subseteq R^d$ and t is a positive real number, the set $\{tx \mid x \in X\}$ will be denoted by tX. Let K_1 and K_2 be convex bodies in R^d that contain the origin as an interior point, and let surface measures m_1 and m_2 be defined on their surfaces S_1 and S_2, respectively. Let $p: S_1 \to S_2$ denote radial projection from the origin. The norm of $x \in R^d$ we shall denote by $|x|$.

THEOREM 1. *Let* K_1 *and* K_2 *be convex bodies in* R^d *containing the origin as an interior point. Let* $f: R^1 \to R^1$ *be a monotonically increasing function. Then*

$$(1) \qquad \frac{1}{m(S_1)} \int_{S_1} f(|x_1|/|p(x_1)|)\,dm_1 \geq \frac{1}{m(S_2)} \int_{S_2} f(|p^{-1}(x_2)|/|x_2|)\,dm_2 .$$

Proof. For convenience, we shall assume that $f(0) = 0$, that f is continuous from the right, and that $m(S_1) = m(S_2)$. Define $g: S_1 \to R^1$ by setting

$$g(x_1) = f(|x_1|/|p(x_1)|).$$

Set $S_1(t) = \{x \mid x \in S_1, g(x) \geq t\}$. In view of Fubini's theorem, it suffices to show that $m(S_1(t)) \geq m[p(S_1(t))]$ for each nonnegative real number t. Now,

$$S_1(t) = \{x_1 \mid x_1 \in S_1, |x_1|/|p(x_1)| \geq t^*\},$$

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