ON GENERATORS OF VON NEUMANN ALGEBRAS

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1. Recently, C. Pearcy and D. Topping [4] and the author [5] separately considered a certain class of von Neumann algebras, and they showed that each algebra of the class can be generated by various small sets of special operators, projections, unitary operators, idempotents, and so forth. The purpose of this paper is to clarify the relation among these results.

2. Throughout the paper, we assume that $\mathbb{H}$ is a separable Hilbert space, and by an *operator* we mean a bounded linear operator on a Hilbert space. A von Neumann algebra $\mathbb{M}$ acting on a Hilbert space is said to be *generated by a family* $\{A, B, \cdots\}$ of operators if $\mathbb{M}$ is the smallest von Neumann algebra containing each member of the family $\{A, B, \cdots\}$, and it is denoted by $\mathbb{M} = R(A, B, \cdots)$. We shall consider von Neumann algebras $\mathbb{M}$ on $\mathbb{H}$ with the property

$$(*) \qquad \mathbb{M} \text{ is } *\text{-isomorphic to } \mathbb{M}_2,$$

where $\mathbb{M}_2$ is the algebra of all 2×2 matrices over $\mathbb{M}$. The following is our result. It sharpens the results in [4] and [5].

THEOREM. *Let $\mathbb{M}$ be a von Neumann algebra with property* (*). *Then the following statements are equivalent.*

(a) $\mathbb{M}$ *has a single generator.*

(b) $\mathbb{M}$ *is generated by one partial isometry.*

(c) $\mathbb{M}$ *is generated by two operators.*

(d) $\mathbb{M}$ *is generated by two unitary operators.*

(e) $\mathbb{M}$ *is generated by three projections.*

3. The essential part of this paper is the proof of the equivalence of (a), (c), and (e).

LEMMA 1. *Suppose that a von Neumann algebra $\mathbb{M}$ is generated by two operators* A *and* B. *Then $\mathbb{M}_2$ is generated by the three operators*

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The proof of this lemma consists of easy computations, which we omit.

LEMMA 2. *Let a von Neumann algebra $\mathbb{M}$ be generated by two operators, and suppose that one of these generators is a normal operator. Then $\mathbb{M}_2$ is generated by a single operator.*

Proof. Let A and B be the generators of $\mathbb{M}$, and suppose that B is normal. We can assume that A is invertible and $\|A\| < 1$. We can also assume that B is an

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