THE DECOMPOSITION OF MATRIX-VALUED MEASURES

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1. INTRODUCTION

By an $m \times n$ - *matrix-valued measure* on a σ-algebra $\mathscr{B}$ over Ω we shall mean a function M from $\mathscr{B}$ into the set of all $m \times n$ matrices over the complex numbers such that for every disjoint sequence of sets $A_1, A_2, \cdots$ in $\mathscr{B}$ with union A, $M(A) = \sum_{k=1}^{\infty} M(A_k)$. A *$\mathscr{B}$-measurable $\ell \times m$-matrix-valued function* on Ω will be a function Φ from Ω into the set of all $\ell \times m$ matrices such that the entries $\Phi_{ij}(\omega) = [\Phi(\omega)]_{ij}$ are $\mathscr{B}$-measurable. We shall define the integral $\int_\Omega \Phi dM$ for a suitable class of $\ell \times m$-matrix-valued functions with respect to an $m \times n$-matrix-valued measure. Such measures and integrals are important in the spectral analysis of multivariate, weakly stationary, stochastic processes (see Masani [8, Section 8]). It was our interest in this subject that led us to the present study, and in Section 7 we shall indicate how our results apply to this theory.

The primary purpose of this paper is to define and study appropriate notions of the total variation of a matricial measure and of the absolute continuity, Radon-Nikodým derivative, and singularity of one matricial measure with respect to another, and to prove matricial versions of the Hahn-Jordan decomposition, Radon-Nikodým theorem, and Lebesgue decomposition. We are able to obtain a reasonably complete theory, but only by renouncing seemingly reasonable definitions. The Hahn-Jordan decomposition of a matrix-valued measure into nonnegative hermitian matrix-valued measures is best viewed not as a finite sum of such measures, but as an integral thereof. Even for a complex-valued measure M, this seems to have been overlooked; but, of course, if M is real-valued, it yields the usual decomposition (Section 4). To get the Radon-Nikodým theorem, we have to define absolute continuity in terms of certain derivatives of the measures rather than in terms of the measures themselves (Section 5). If two matrix-valued functions are equal almost everywhere with respect to a matrix-valued measure M, and if N is absolutely continuous with respect to M, it does not necessarily follow that the functions are equal almost everywhere with respect to N. This is because matrix multiplication is not commutative. Unless we exercise great care in the definition of Radon-Nikodým derivatives, the Radon-Nikodým derivatives of two equivalent measures will not always turn out to be inverses of one another. Here the notion of generalized inverse due to Penrose [10] is very useful (Section 2). The usual notion of the carrier of a measure must be supplanted by that of a projection matrix-valued function, which is the matricial analogue of the indicator function (Section 6). There is not just a single Lebesgue decomposition of one measure with respect to another, but to each measure of a certain class there corresponds a distinct decomposition (Section 6). This has helped to clarify certain problems arising in the orthogonal decomposition of stochastic processes (Section 7).

Hardly any work seems to have been done on the problem of obtaining a Radon-Nikodým derivative and a Lebesgue decomposition of one operator-valued measure with respect to another. Even for vector-valued measures, the literature is scanty

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