ORDINAL INVARIANTS FOR TOPOLOGICAL SPACES

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0. INTRODUCTION

Cardinal invariants such as weight, density, and dimension have been widely used in the classification of topological spaces. More rarely (see for example Maurice [10] and Stone [13]), ordinal invariants have been employed. In this paper, we introduce two related ordinal invariants, σ and κ, first in the categories of sequential and k-spaces (Section 1) and later in arbitrary spaces (Section 6). (For an informed opinion of the importance of the category of k-spaces, see Steenrod [12]). Our main result is the existence, for each $\alpha \leq \omega_1$, of a countable, zero-dimensional Hausdorff space X with $\sigma(X) = \kappa(X) = \alpha$ (Theorems 4.1 and 5.1).

1. PRELIMINARIES

A topological space X is a *k-space* (see Arhangel'skiĭ [2], Cohen [4], and Steenrod [12]) if a subset F of X is closed whenever its intersection with each bicompact subset K of X is closed in K. For each subset A of X, we shall write $x \in A^{\sim}$ if and only if $x \in \mathrm{cl}_K(A \cap K)$ for some bicompact subset K of X. Now let $A^0 = A$, and for each nonlimit ordinal $\alpha = \beta + 1$, let $A^\alpha = (A^\beta)^\sim$. If α is a limit ordinal, let $A^\alpha = \bigcup\{A^\beta \mid \beta < \alpha\}$. For an arbitrary space X, let $\kappa(X)$ denote the infimum of the ordinals α such that $A^\alpha = \mathrm{cl}_X A$ for each subset A of X. A straightforward argument, involving only cardinality in one direction and the fact that a single point may be added to a bicompact set without destroying bicompactness, establishes the following result.

1.1. PROPOSITION. *X is a k-space if and only if $\kappa(X)$ exists.*

Since the definition of k-spaces was given in terms of closure only, the following proposition is obvious.

1.2. PROPOSITION. *κ is a topological invariant in the category of k-spaces.*

$\kappa(X) = 0$ if and only if X is discrete, and $\kappa(X) \leq 1$ is precisely the criterion that determines the k'-spaces (see Arhangel'skiĭ [2]).

We now restrict our attention to a special case. A subset U of a topological space X is *sequentially open* if each sequence converging to a point in U is eventually in U. The space X is *sequential* if each sequentially open subset of X is open (see Franklin [8], [9]). For each subset A of X, we shall denote by $A^\wedge$ the set of all limits of sequences in A. Now let $A^0 = A$, and for each ordinal $\alpha = \beta + 1$, let $A^\alpha = (A^\beta)^\wedge$. If α is a limit ordinal, let $A^\alpha = \bigcup\{A^\beta \mid \beta < \alpha\}$. (Whether A^α refers to the sequential closure $^\wedge$ or the k-closure $^\sim$ will always be clear from the context.) Denote by $\sigma(X)$ the infimum of the ordinals α with the property that $A^\alpha = \mathrm{cl}_X A$ for all $A \subseteq X$. The following is a folk theorem.

1.3. PROPOSITION. *X is sequential if and only if $\sigma(X)$ exists. In this case,* $\sigma(X) \leq \omega_1$ (where ω_1 is the first uncountable ordinal).

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