COMPLETENESS OF $\{A \sin nx + B \cos nx\}$ ON $[0, \pi]$

Robert Feinerman and D. J. Newman

The classical L^2-theory of Fourier series tells us that on $[0, \pi]$ the sines are complete while the cosines are not (unless we include $1 = \cos 0x$). It is natural to ask the completeness question about the family $\{A \sin nx + B \cos nx\}$ (A and B arbitrary complex numbers), and indeed, to generalize to the other L^p-spaces. The question is also interesting in the case $p = \infty$, if here we consider instead of L^∞ (which is not separable) its subspace C of functions continuous on $[0, \pi]$. In this paper we give the complete answers to these questions.

These answers are most simply expressed in terms of a slightly different notation. Observe that if $A/B = \pm i$, we are looking at the set $\{e^{inx}\}$ or $\{e^{-inx}\}$, and that in this case completeness holds in the strongest topology of all, namely in $C[0, \pi]$ (and even in $C[0, \tau]$ for any $\tau < 2\pi$). If $A/B \neq \pm i$, we can write

$$A \sin nx + B \cos nx = \pm\sqrt{A^2 + B^2} \sin\left(nx + \frac{\pi}{2}\alpha\right),$$

where $-1 \leq \Re\alpha \leq 1$. Also, since the replacement of x by $\pi - x$ shows that completeness for α is equivalent to that for $-\alpha$, we impose the further restriction $0 \leq \Re\alpha \leq 1$. In all that follows, we shall assume this, and we shall denote by S_α the set of functions $\left\{\sin\left(nx + \frac{\pi}{2}\alpha\right)\right\}$ $(n = 1, 2, \cdots)$; also, we abbreviate $L^p[0, \pi]$ to L^p.

THEOREM 1. I. S_α *is complete in* $L^1 \Leftrightarrow \Re\alpha \neq 1$.

II. *Let* $1 < p < \infty$. S_α *is complete in* $L^p \Leftrightarrow \Re\alpha \leq 1/p$.

III. S_α *is complete in* C $\Leftrightarrow \Re\alpha = 0$, $\alpha \neq 0$.

In the α-notation, the completeness set for L^p is a strip, while for C it is the imaginary axis excluding the origin. If we map back to the B/A notation, these sets are "lens shaped." For $1 < p < \infty$, the completeness set in the B/A-plane consists of the inside and boundary of the curve formed by two circular arcs, each passing through $\pm i$ and making the interior angle π/p with the imaginary axis. (When $p = 2$, this set becomes the closed unit disc.)

For L^1, the completeness set is the entire plane except for the points iy on the imaginary axis with $|y| > 1$.

For C, the set consists of the points iy on the imaginary axis with $0 < |y| \leq 1$.

A strange situation exists in the case where B/A is imaginary. Here our theorem tells us that $\{\sin nx + i\lambda \cos nx\}$ is complete in the strongest sense (in C) if $0 < \lambda \leq 1$, and that for $\lambda > 1$ the family is not even complete in L^1.

If instead of letting n range through the positive integers, we throw in also $n = 0$, then completeness is essentially universal.

THEOREM 2. *If* $\alpha \neq 0$, *the set of functions* $1 \cup S_\alpha$ *is complete in* C.

Received May 29, 1967.
This research was partially supported by NSF Grant GP-4391.