DOUBLY GENERATED FUCHSIAN GROUPS

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In Memory of H. Mirkil

INTRODUCTION

Our main theorem concerns geodesic polygons in hyperbolic geometry; it gives a necessary condition for certain finite sets of real two-by-two matrices to generate discrete groups. We suppose that we are given a compact polygon with a finite, even number of sides, and we suppose that the sides are matched in pairs by linear fractional transformations satisfying an orientation condition. If the group generated by these transformations is discrete, then the theorem is that the translates by the group of this polygon cover most of the points in the hyperbolic plane the same number of times.

This result is stated more precisely and proved in Section 1. The rest of the paper illustrates the theorem by applying it to the following question: When is a doubly generated group of analytic automorphisms of the upper half-plane discrete? The theorem of Section 1 will be used to answer the question for the case where neither generator is hyperbolic. The nature of the criterion we give is that one need only find whether the trace of the product of a well-determined power of one generator by a power of the other generator appears in a list. For example, if both generators are elliptic, the list contains two classes of major cases corresponding to classical presentations of triangle groups, and it contains five classes of exceptional cases corresponding to some geometric configurations discussed at the beginning of Section 3 (see Theorem 2.3).

The polygon theorem is given in Section 1, the criterion for groups with two eliptic generators is stated in Section 2 and proved in Section 3, and doubly generated groups at least one of whose generators is parabolic are treated in Section 4. I wish to thank B. Maskit for suggesting the problem solved by Theorem 2.3 and for his advice connected with these results, and I wish to thank the referee for some improvements in the exposition.

1. POLYGON THEOREM

Throughout this section, U denotes the upper half-plane with the hyperbolic geometry. The hyperbolic area of a set X is m(X), the boundary of X is ∂X, and the image of X under a set D of transformations of U is D(X).

THEOREM 1.1. *Let* P *be an open, simple, finite-sided geodesic polygon in* U *whose closure* $\overline{P}$ *in* U *is compact. Suppose that all the sides of* P *are matched in disjoint pairs by analytic automorphisms of* U. *Say,* $L_a(s_a) = s_{a'}$. *As an orientation condition on the transformations* L_a, *suppose that for each* x *within the side* s_a, *each sufficiently small disc* N *centered at* x *satisfies the inclusion condition*

$$N \subseteq s_a \cup P \cup L_a^{-1}(P).$$

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